

**Complexity of Computing Topological Degree of
Lipschitz Functions in N Dimensions**

T. BOULT*

K. SIKORSKI**

Columbia University*

University of Utah**

CUCS-188-85

*This work supported in part by DARPA grant N00039-84-C-0165 and in part by NSF grant MCS-782-3676

Abstract We find lower and upper bounds on the complexity $\text{comp}(\text{deg})$, of computing the topological degree of n functions defined on the n -dimensional unit cube C^n , $f : C^n \rightarrow R^n$, $n \geq 2$, which satisfy a Lipschitz condition with constant K and whose infinity norm at each point on the boundary of C^n is at least d , $d > 0$, and such that $\frac{K}{8d} \geq 1$.

A lower bound, $\text{comp}_{\text{low}} \simeq 2n(\frac{K}{8d})^{n-1}(c+n)$ is obtained for $\text{comp}(\text{deg})$, assuming that each function evaluation costs c and elementary arithmetic operations and comparisons cost unity.

We prove that the topological degree can be computed using $A = (\lfloor \frac{K}{2d} + 1 \rfloor + 1)^n - (\lfloor \frac{K}{2d} + 1 \rfloor - 1)^n$ function evaluations. It can be done by an algorithm φ^* due to Kearfott, with cost $\text{comp}(\varphi^*) \cong n(c + \frac{n^2}{2}(n-1)!)$. Thus for small n , say $n \leq 5$, and small $\frac{K}{2d}$, say $\frac{K}{2d} \leq 9$, the degree can be computed in time at most $10^5(c+300)$. For large n and/or large $\frac{K}{2d}$ the problem is intractable.

1. INTRODUCTION.

The problem of computing the topological degree of a function has been studied in many recent papers, see Kearfott (1977,1979), Stenger (1975), and Stynes (1979a,1979b,1981). From the topological degree one may ascertain whether there exists a zero of a function inside a domain. Namely, Kronecker's theorem, see Ortega and Rheinboldt (1970), states that if the degree is not zero, then there exists at least one zero of a function inside the domain. By computing a sequence of domains within nonzero degrees with decreasing diameters one can obtain a region with arbitrarily small diameter which contains at least one zero of the function, see Kearfott (1977,1979) and Stynes (1981). Algorithms proposed in these papers were tested by their authors on relatively easy examples. They concluded that the degree of an arbitrary continuous function could be computed. It was observed, however, see Kearfott (1977,1979) and Stynes (1981), that the algorithms may require an unbounded number of function evaluations.

In this paper we restrict the class of functions, this enables us to compute the degree for every element in the restricted class using an a priori bounded number of function evaluations. We consider the class F of Lipschitz functions with constant K , defined on the unit cube $C^m \subset R^n$, $f: C^m \rightarrow R^n$, such that $\|f(x)\|_\infty \geq d > 0$, for every $x \in \partial C^m$, the boundary of C^m , and $\frac{K}{8d} \geq 1$. Note that if $\frac{K}{2d} < 1$ then the functions in F do not have zeros and therefore the degree is zero for every f . The case $1 \leq \frac{K}{2d} < 4$ is open. The information on f , $N_m(f)$, consists of m values of f on ∂C^m which may be computed adaptively. This form of information is assumed since the topological degree is defined by the values of f on ∂C^m , see Ortega and Rheinboldt (1970). The topological degree is computed by means of an algorithm φ which is a mapping depending on the information, $\varphi: N_m(F) \rightarrow I$, where I denotes the set of all integers.

In this paper we solve the following problems:

- (1.1) We exhibit information N_m^* which uniquely determines the degree of f for every $f \in F$.

This information consists of

$$A = \left(\left\lfloor \frac{K}{2d} + 1 \right\rfloor + 1 \right)^n - \left(\left\lfloor \frac{K}{2d} + 1 \right\rfloor - 1 \right)^n$$

function evaluations, see Sect. 3.

- (1.2) We exhibit an algorithm φ^* due to Kearfott (1979) which uses N_m^* to compute the degree, see Sect. 4.
- (1.3) We find a lower bound m^* , roughly equal $2n(\lfloor \frac{K}{8d} \rfloor)^{n-1}$, on the number of function evaluations necessary to find the degree of f for every f in F using arbitrary information N_m , see Sect. 5.

We remark that information N_m^* is *parallel (nonadaptive)*, i.e., the evaluation points are given a priori. Thus N_m^* can be efficiently implemented on a parallel computer yielding an almost optimal speed-up, see Traub and Woźniakowski (1984) for further discussion.

Assuming that each function evaluation costs c and elementary operations cost unity, (1.1) yields a lower bound comp_{low} on the complexity, $\text{comp}(\text{deg})$, of the problem

$$\text{comp}_{\text{low}} \simeq 2n \left(\frac{K}{8d} \right)^{n-1} (c + n)$$

If $\frac{K}{8d}$ is large and/or n is large then the lower bound is so huge that the problem is intractable. For example take $\frac{K}{8d} = 10^3$ and $n = 10$ then the $\text{comp}_{\text{low}} \simeq 2 \cdot 10^{28} (c + 10)$.

The cost of algorithm φ^* is roughly $A(c + \frac{n^2}{2}(n-1)!)$. Thus for small n , say $n \leq 5$ and small $\frac{K}{2d}$, say $\frac{K}{2d} < 10$, φ^* computes the degree in time at most roughly $10^5 \cdot (c + 300)$.

We remark that in Boulton and Sikorski (1985a) (see also Boulton (1986)) we find the complexity $\text{comp}_2(\text{deg})$ for the two dimensional case,

$$(1.4) \quad \text{comp}_2(\text{deg}) = 4 \left\lfloor \frac{K}{4d} \right\rfloor (c + a) - 1$$

where $a \in [2, 24]$.

In Boulton and Sikorski (1985a) we exhibit an algorithm with cost as (1.4) with $a=24$. This algorithm ($n = 2$) as well as the n -dimensional algorithm φ^* (for small n , $n \geq 3$) exhibited here are implemented in Boulton and Sikorski (1985b), see also Boulton (1986).

2. BASIC DEFINITIONS

Let $C^n = [0, 1]^n$ be the unit cube in R^n , $n \geq 2$, I the set of all integers, $\|\cdot\| = \|\cdot\|_\infty$ the infinity norm in R^n and $\theta = (0, \dots, 0) \in R^n$. For a given positive d and K define

$$(2.1) \quad F = \left\{ f : C^n \rightarrow R^n, f = (f_1, \dots, f_n), \|f(x) - f(y)\| \leq K\|x - y\| \right. \\ \left. , \forall x, y \in C^n \text{ and } \|f(x)\| \geq d, \forall x \in \partial C^n, \text{ and } \frac{K}{8d} \geq 1 \right\}$$

Our problem is to find the topological degree, $\deg(f, C^n, \theta)$ of f relative to C^n at θ , see Ortega and Rheinboldt (1970), for every f in F . To solve this problem we use *information* N_m and an algorithm φ using N_m . These are defined as in Traub and Woźniakowski (1980): Let $f \in F$ and

$$(2.2) \quad N_m(f) = [f(x_1), \dots, f(x_m)]$$

where $x_1 \in \partial C^n$ is given a priori, $x_j = \bar{x}_j(f(x_1), \dots, f(x_{j-1}))$ and \bar{x}_j is a transformation $\bar{x}_j : R^{n \cdot (j-1)} \rightarrow \partial C^n$, $j = 2, \dots, m$. If \bar{x}_j are constant, i.e. all x_j are given a priori, then the information is called *parallel (nonadaptive)*, otherwise it is called *sequential (adaptive)*.

By *minimal cardinality number* m_{\min} we mean the minimal m for which there exists information N_m which uniquely determines the degree of any f in F , i.e.

$$N_m(f) = N_m(f') \Rightarrow \deg(f', C^n, \theta) = \deg(f, C^n, \theta), \forall f, f' \in F.$$

Knowing N_m we approximate $\deg(f, C^n, \theta)$ by an algorithm φ , which is an arbitrary mapping

$$(2.3) \quad \varphi : N_m(F) \rightarrow I.$$

We exhibit an algorithm φ^* using information N_m^* (mentioned in the Introduction) which was developed by Kearfott (1979) and is based on his parity theorem.

3. INFORMATION N_A^*

In this section we prove that the computation of function values on a uniform grid with diameter less than $2\frac{d}{K}$ uniquely determines the degree.

Namely let $M = \lfloor \frac{K}{2d} + 1 \rfloor$ and $R = 1/M$. Subdivide each $(n-1)$ face of C^n into M^{n-1} equal cubes of diameter R , by subdividing each edge into M equal intervals of length R . In this way we obtain a subdivision of ∂C^n into $2nM^{n-1}$ cubes C_i of diameter R :

$$(3.1) \quad \partial C^n = \bigcup_{i=1}^{2nM^{n-1}} C_i.$$

Let $X = \{x_1, \dots, x_A\}$ be the set of all vertices of cubes C_i . Observe that

$$A = (M+1)^n - (M-1)^n.$$

Then define the information operator

$$N_A^* = [f(x_1), \dots, f(x_A)], \quad \forall f \in F.$$

We show

LEMMA 3.1. *The information N_A^* uniquely determines the degree for every f in F , i.e.*

$$N_A^*(f) = N_A^*(g) \text{ implies } \deg(f, C^n, \theta) = \deg(g, C^n, \theta), \quad \forall f, g \in F.$$

■

PROOF: To prove Lemma 3.1 we use the Poincaré-Bohl Theorem, see Ortega and Rheinboldt (1970). Namely let $h(t, z) = tf(z) + (1-t)g(z), \forall t \in [0, 1]$ and $\forall z \in \partial C^n$. To conclude that $\deg(f, C^n, \theta) = \deg(g, C^n, \theta), \forall f, g \in F$ such that $N_A^*(f) = N_A^*(g)$, it is enough to show that the homotopy $h(t, z)$ is non zero for every $t \in [0, 1]$ and every $z \in \partial C^n$. To show this take an arbitrary $z \in \partial C^n$. Then there exists an x_j such that

$\|x_j - z\| \leq R/2 < \frac{d}{K}$. Since $x_j \in \partial C^m$ and $f \in F$ we get $\|f(x_j)\| = |f_i(x_j)| \geq d$ for some $i, 1 \leq i \leq n$. Then we have $|f_i(z) - f_i(x_j)| \leq \|f(z) - f(x_j)\| \leq K\|z - x_j\| < d$. This implies that $f_i(z) \neq 0$ and $\text{sign } f_i(z) = \text{sign } f_i(x_j)$. Since $f(x_j) = g(x_j)$ and $g \in F$, then $g_i(z) \neq 0$ and $\text{sign } g_i(z) = \text{sign } f_i(z)$. Therefore for every $t \in [0, 1]$ we have

$$\begin{aligned} \|h(t, z)\| &\geq |tf_i(z) + (1-t)g_i(z)| \\ &= t|f_i(z)| + (1-t)|g_i(z)| \\ &\geq \min(|f_i(z)|, |g_i(z)|) > 0 \end{aligned}$$

which completes our proof. ■

4. ALGORITHM USING INFORMATION N_A^*

We exhibit here an algorithm φ^* , due to Kearfott (1979), using the information N_A^* to compute the degree. The algorithm φ^* and information N_A^* are implemented in a Fortran subroutine in Boulton and Sikorski (1985b), where a number of numerical tests are also reported. Fortran Code can be found in the appendices of Boulton (1986).

First we show that the evaluation points $x_i, i = 1, \dots, A$, yield an *impartial refinement* of ∂C relative to the sign of f , for every f in F .

Impartial refinement, see Stynes (1979a), is defined as follows:

Definition 4.1: If $n = 1$ then $\partial[0, 1] = \{0\} \cup \{1\}$ is impartially refined relative to sign of f iff $f(0) \cdot f(1) < 0$.

If $n > 1$ then ∂C^m is impartially refined relative to the sign of f iff ∂C^m may be written as a union of a finite number of $(n-1)$ regions β_1, \dots, β_q (by an $(n-1)$ region we mean a union of a finite number of $(n-1)$ simplices) in such a way that:

(4.1) the $(n-1)$ - interiors of $\beta_i - s$ are pairwise disjoint;

(4.2) $\forall i \in [1, \dots, q], \exists r_i \in [1, \dots, n] : f_{r_i}$ is of constant sign on β_i ;

(4.3) if $\beta_i \cap \beta_j \neq \emptyset$ for $i \neq j$ then $r_i \neq r_j$,

(4.4) if S_i is an $(n - 1)$ simplex in β_i such that S_i has an $(n - 2)$ - face in $\partial\beta_i$
then this face is also an $(n - 2)$ face of some $(n - 1)$ simplex S_j in $\beta_j, i \neq j$.

■

Now consider the subdivision (3.1) of ∂C^n into $2nM^{n-1}(n - 1)$ -cubes, and subdivide each of the cubes C_i into $(n - 1)!$ $(n - 1)$ simplices as described in Jeppson (1972). This forms a simplicial subdivision of ∂C^n , see Allgower et. al. (1971) and Jeppson (1972), into $2nM^{n-1}(n - 1)!$ $(n - 1)$ -simplices:

$$(4.5) \quad \partial C^n = \sum_{j=i}^L t_j S_j, \quad t_j = \pm 1, \quad L = 2n!M^{n-1},$$

where S_j are oriented, see Kearfott (1979), and Stynes (1979a,1979b,1981). From, $(n - 1)$ -simplices. Note that the vertices of S_j are uniquely determined by this subdivision and the evaluation points z_i . The explicit formulas for the vertices of S_j 's are given by Allgower et. al. (1971) Jeppson (1972) and fortran code generating them can be found in Boulton and Sikorski (1985) and Boulton (1986).

We are now ready to prove:

LEMMA 4.1. *The subdivision (4.5) yields an impartial refinement of ∂C^n relative to the sign of f , for every f in F .* ■

PROOF: We construct the regions β_i from Def. 4.1. For an arbitrary f in F and for each cube C_i in the subdivision (3.1) choose a component f_{j_i} of f which is of constant sign on C_i . Such a component exists since for some $j_i, |f_{j_i}(z_i)| \geq d$ where z_i is the center of C_i . Thus f_{j_i} is of constant sign on C_i since the radius of C_i is less than $\frac{d}{K}$ and f is in F , i.e. $|f_{j_i}(z) - f_{j_i}(z_i)| \leq \|f(z) - f(z_i)\| \leq K\|z - z_i\| < d$ for $\|z - z_i\| < \frac{d}{K}$, which yields $\text{sgn}(f_{j_i}(z)) = \text{sgn}(f_{j_i}(z_i))$. Then group the cubes C_i to form connected

regions $\beta_{j,1}, \dots, \beta_{j,k_j}$ such that f_j is of constant sign on each $\beta_{j,l}, l = 1, \dots, k_j$, and $\beta_{j,l_1} \cap \beta_{j,l_2} = \emptyset, l_1 \neq l_2$. In this way we obtain a decomposition of ∂C^n

$$(4.6) \quad \partial C^n = \bigcup_{j=1}^n \bigcup_{l=1}^{k_j} \beta_{j,l},$$

which satisfies (4.1)-(4.3) of Def. 4.4. Since each cube in every $\beta_{j,l}$ is subdivided into $(n-1)$ simplices forming a simplicial subdivision of ∂C then (4.4) of definition (4.1) is also met. This completes the proof. ■

Remark 4.1: Since the impartial refinement of ∂C^n is also a sufficient refinement (see Kearfott (1979) and Stynes (1979a, 1979b, 1981) for the definition of sufficient refinement, and Stynes (1979a, Th. 3.3) for the above result) then we can use Kearfott's Parity theorem, see Kearfott (1979), to compute the degree. ■

Let $S = [S_1, \dots, S_n]$ be an $(n-1)$ simplex in R^n with vertices $S_i, i = 1, \dots, n$. The range matrix $R(S, f)$ associated with S and $f \in F$ is an $n \times n$ matrix:

$$R(S, f) = [r_{i,j}]_{i,j=1}^n, \quad r_{i,j} = \text{sgn}(f_j(S_i)),$$

where

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The range matrix $R(S, f)$ is feasible if

$$(4.7) \quad \begin{aligned} r_{ij} &= 1, \quad \forall i \geq j \quad \text{and} \\ r_{i,i+1} &= 0, \quad i = 1, \dots, n-1. \end{aligned}$$

Define the parity of the range matrix $R(S, f)$ by

$$\text{Par}(R(S, f)) = \begin{cases} 1 & \text{if } R(S, f) \text{ is feasible after an even permutation of rows} \\ -1 & \text{if } R(S, f) \text{ is feasible after an odd permutation of rows;} \\ 0 & \text{otherwise.} \end{cases}$$

We remark that the parity can be computed by roughly $n^2/2$ comparisons.

Define the algorithm φ^* using N_A^* by

$$(4.8) \quad \varphi^*(N_A^*(f)) = \sum_{j=1}^L \text{Par}(R(t_j \cdot S_j, f)),$$

where L and t_j, S_j are as in (4.5). Then Remark 4.1 and Parity Theorem, see Kearfott (1979), imply that

$$\text{deg}(f, C^n, \theta) = \varphi^*(N_A^*(f)), \quad \forall f \in F.$$

Observe that implementation of φ^* requires computing the parities of $L = 2nM^{n-1}(n-1)!$ simplices. Thus the complexity of φ^* is at most

$$\text{comp}(\varphi^*) \leq Ac + 2nM^{n-1} \cdot \frac{n^2}{2}(n-1)! \leq A \left(c + \frac{n^2}{2}(n-1)! \right)$$

where c is the cost of one function evaluation and arithmetic operations and comparisons cost unity.

5. A LOWER BOUND

In this section we find a lower bound on the number of function evaluations needed to compute the topological degree of functions from the class F .

THEOREM 5.1. *For any information N_m , with $m \leq 2n \lfloor \frac{K}{8d} \rfloor^{n-1} - 1$, there exists two functions f^* , f^{**} in F such that $N_m(f^{**}) = N_n(f^*)$ and $|\deg(f^{**}, C^n, \theta)| = 1$ and $\deg(f^{**}, C^n, \theta) = 0$.* ■

Note that Theorem 5.1 implies (1.3), i.e. to compute the degree for any $f \in F$ using arbitrary information N_m we must use at least $m = 2n \lfloor \frac{K}{8d} \rfloor^{n-1}$ function evaluations. This lower bound is exponential in the dimension n , thus for large n and/or large $\frac{K}{8d}$ the problem is intractable.

In order to prove Theorem 5.1 we need the following lemma.

LEMMA 5.1. *Let H^n be an n -cube in C^n with diameter $8\frac{d}{K} \leq 1$ such that:*

$$(5.1) \quad \begin{aligned} B^n = H^n \cap \partial C^n \text{ is an } b(n-1) \text{ face of } H^n, \text{ and corresponding} \\ (n-1) \text{ faces of } H^n \text{ and } C^n \text{ are parallel.} \end{aligned}$$

Then there exist a function $f^n \in F$, $f^n = (f_1^n, \dots, f_n^n)$, such that:

$$(5.2) \quad \text{there exists exactly one zero } \alpha^n \text{ of } f^n, \|\alpha^n - b^n\| = \frac{d}{K},$$

$$\text{where } b^n \text{ is the center of } B^n, \text{ and } \text{dist}(\alpha^n, B^n) = \frac{d}{K}$$

$$(5.3) \quad f_j^n(z) = d \text{ for } z \in C^n - H^n, \quad \forall_j,$$

$$(5.4) \quad \|f^n(z)\| = d \text{ for } z \in \partial C^n,$$

$$(5.5) \quad \left. \frac{\partial f_j^n}{\partial z_i} \right|_{\alpha^n} = \pm K \delta_{ij} \text{ where } \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

which implies that α is a simple zero,

$$(5.6) \quad -d \leq f_j^n(z) \leq d, \quad \forall z \in C^n, \quad \forall j.$$

$$(5.7) \quad \forall z \in C^n : \|z - b^n\| \geq 2\frac{d}{K}, \quad \exists j : f_j^n(z) = d.$$

■

PROOF: The proof is by induction on n . Let $n = 2$ and let H^2 be a square satisfying 5(.1). Without loss of generality assume that $B^2 \subset [0, 1]$, $B^2 = [b_1, b_2]$, so $b^2 = (b_1 + b_2)/2, 0$. Let $c_1 = b^2 + (\frac{d}{K}, \frac{d}{K})$ and $c_2 = b^2 - (\frac{d}{K}, \frac{d}{K})$. Define the function $f^2 : C^2 \rightarrow R^2$ by:

$$f^2(z) = (f_1^2(z), f_2^2(z))$$

$$f_1^2(z) = \min(d, \max(-d, -2d + K\|z - c_1\|)),$$

$$f_2^2(z) = \min(d, \max(-d, -2d + K\|z - c_2\|)),$$

see Fig. 5.1.

Observe that f^2 satisfies Lipschitz condition with constant K and that $\alpha^2 = b^2 + (-\frac{d}{K}, \frac{d}{K})$ is the unique zero of f^2 . Thus $\text{dist}(\alpha^2, B^2) = \frac{d}{K}$ and $\|\alpha^2 - b^2\| = \frac{d}{K}$, which implies (5.2). The definition of f^2 directly yields (5.3), (5.4), (5.6), and (5.7). For (5.5) observe that

$$\left. \frac{\partial f_1^2}{\partial z_i} \right|_{\alpha^2} = K\delta_{i,1} \text{ and } \left. \frac{\partial f_2^2}{\partial z_i} \right|_{\alpha^2} = K\delta_{i,2}, \quad i = 1, 2.$$

Thus the lemma holds for $n = 2$.

INDUCTION STEP.

Now assume that Lemma 5.1 holds for $n - 1$. Let $H^n \subset C^n$, $\text{diam}(H_n) = 8\frac{d}{K}$, be an n -cube such that (5.1) holds. Without loss of generality assume that all points in B^n have the j -th ($j \neq n$) component equal to 1. (If $j = n$ then the same construction follows with the n -th dimension replaced by the 1-st dimension).

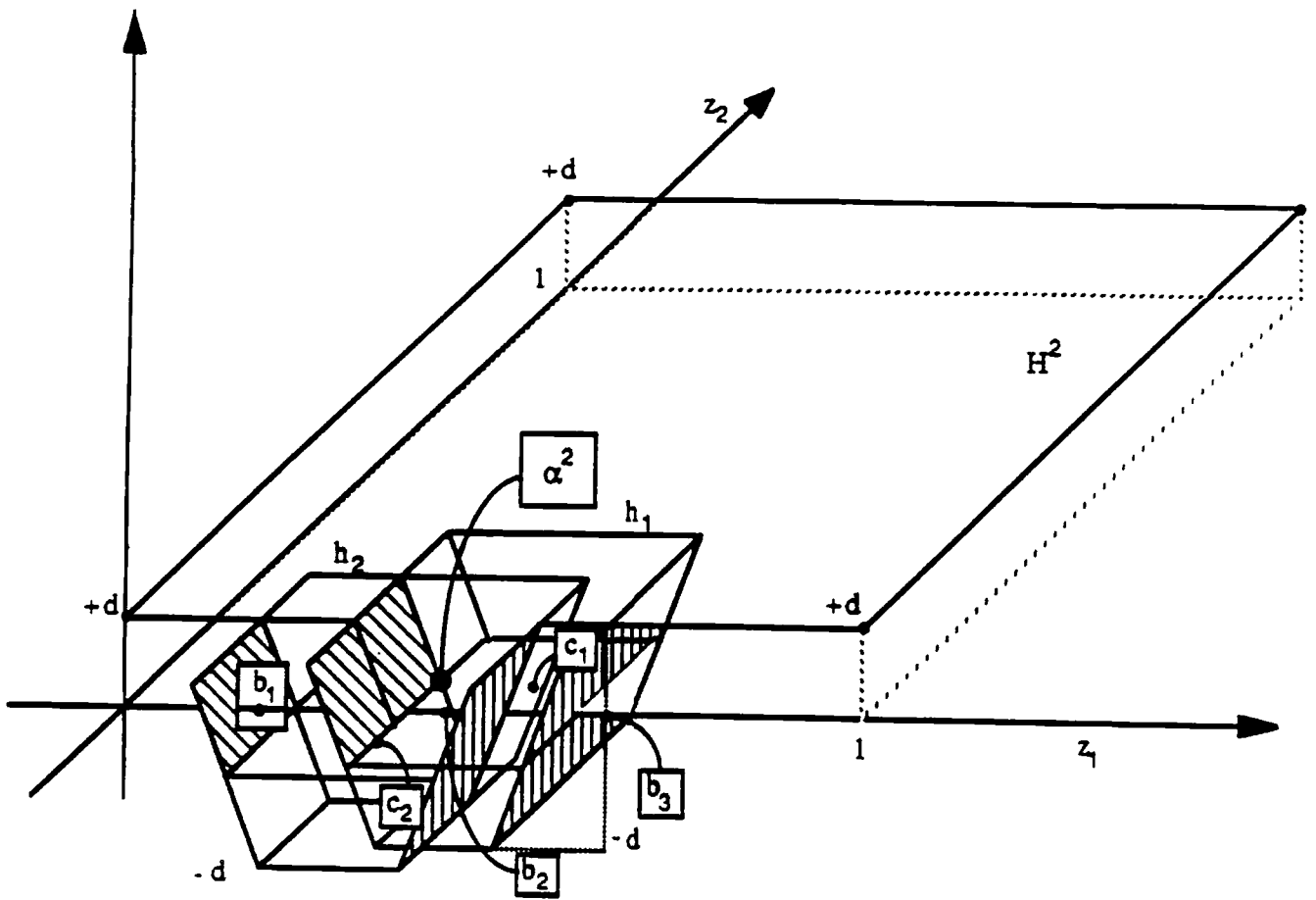


Figure 5.1

Let H^{n-1} be the orthogonal projection of H^n onto the C^{n-1} . From the induction assumption there exists f^{n-1} for H^{n-1} such that (5.1)-(5.6) hold. Define. (see Fig. 5.2),

$$(5.8) \quad \alpha^n = (\alpha_1^n, \dots, \alpha_n^n),$$

where $\alpha_j^n = \alpha_j^{n-1}$, $j = 1, \dots, n-1$, $\alpha_n^n = b_n^n - \frac{d}{K}$ and $b^n = (b_1^n, \dots, b_n^n)$ is the center of B^n .

Let

$$(5.9) \quad y(i_1, \dots, i_{n-2}) = \left(b_1^n + i_1 \cdot \frac{d}{K}, \dots, b_{j-1}^n + i_{j-1} \cdot \frac{d}{K}, 1, b_{j+1}^n + i_{j+1} \cdot \frac{d}{K}, \dots, b_n^n + \frac{d}{K} \right)$$

where $i_j \in \{+1, -1\}$, $j = 1, \dots, n-2$, i.e. these are 2^{n-2} points in B^n .

Define the function $g^n, g^n : C^n \rightarrow R$, by

$$(5.10) \quad g^n(z) = \min(d, \bar{y}_1(z), \dots, \bar{y}_{n-2}(z)),$$

where $\bar{y}_i(z) = \max(-d, -2d, +K\|z - y_i\|)$, and $y_i, i = 1, \dots, 2^{n-2}$, are all of the points $y(i_1, \dots, i_{n-2})$. Observe that g^n satisfies Lipschitz condition with constant K as minima and maxima of such functions. Also note that the zero set of g^n , $Z_0 = \{z \in C^n : g^n(z) = 0\}$, (see Figure 5.2 for the case with $n = 3$) is given by

$$Z_0 = \{z \in C^n : \exists i : 2 \frac{d}{K} = \|z - y_i\| \leq \|z - y_j\|, j = 1, \dots, 2^{n-2}\}.$$

Finally for $z \in C^n$, $z = (z_1, \dots, z_n)$, let $\bar{z} = (z_1, \dots, z_{n-1})$ be the orthogonal projection of z onto C^{n-1} .

Define

$$f^n(z) = (f_1^n(z), \dots, f_n^n(z)),$$

where

$$f_n^n(z) = \begin{cases} g^n(z), & \forall z \in H^n \\ d, & \forall z \in C^n - H^n, \end{cases}$$

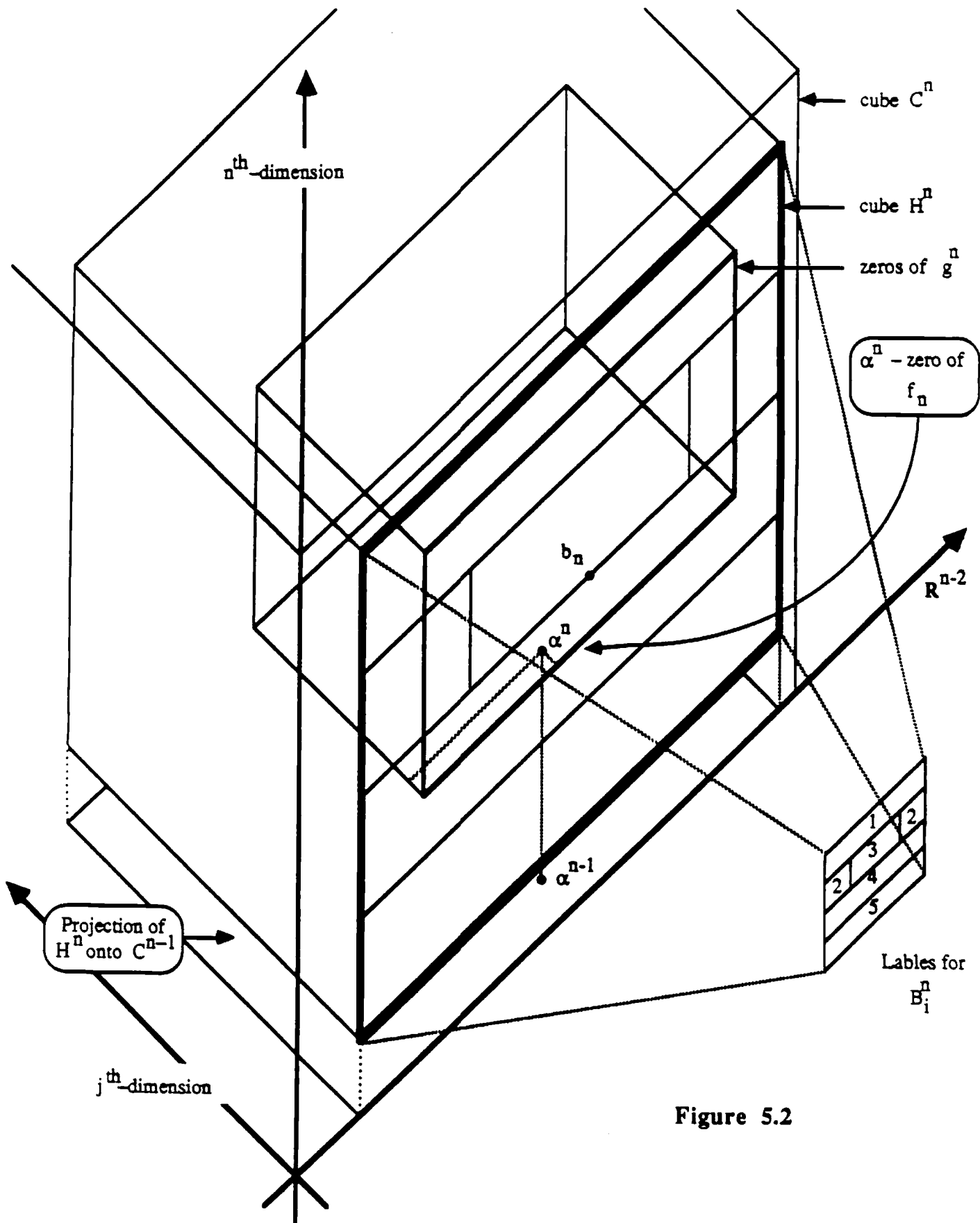


Figure 5.2

and

$$f_i^n(z) = \begin{cases} \min(d, \max(f_i^{n-1}(\bar{z}), -d + K|z_n - (b_n^n - \frac{2d}{K}))), & \text{for } z \in H^n : \\ (f_i^{n-1}(\bar{z}), & b_n^n - 4\frac{d}{K} \leq z_n \leq b_n^n - 2\frac{d}{K}, \\ \min(d, \max(f_i^{n-1}(\bar{z}), -d + K|z_n - b_n^n)), & \text{for } z \in H^n : \\ d & b_n^n - 2\frac{d}{K} < z_n < b_n^n; \\ & \text{for } z \in H^n : \\ & b_n^n \leq z_n \leq b_n^n + \frac{4d}{K}; \\ & \text{for } z \in C^n - H^n, \end{cases}$$

for $i = 1, 2, \dots, n-1$.

Now we show that f^n is in F and satisfies (5.2)-(5.7).

First we check that f^n is continuous. Since for every $z \in C^n - \text{Int}(H^n)$, $\|z - b^n\| \geq 4\frac{d}{K}$, then $\bar{y}_j(z) \geq -2d + K\|z - y_i\| \geq -2d + K \cdot 3\frac{d}{K} = d \forall j = 1, \dots, 2^{n-2}$, and therefore $g^n(z) = d$. This and continuity of g^n implies that f_n^n is continuous. Thus we must only check the continuity of f_j^n , $j = 1, \dots, n-1$, at all $z \in C^n - H^n$ and $z \in H^n$ such that $z_n = b_n^n$ or $z_n = b_n^n - 2\frac{d}{K}$. First let $z \in C^n - H^n$. If $z_n = b_n^n - 4\frac{d}{K}$ then $f_i^n(z) = \min(d, \max(f_i^{n-1}(\bar{z}), d)) = d$. If $z_n = b_n^n + 4\frac{d}{K}$ then $f_i^n(z) = \min(d, \max(f_i^{n-1}(\bar{z}), 3d)) = d$. If $b_n^n - 4\frac{d}{K} < z_n < b_n^n + 4\frac{d}{K}$ then $\bar{z} \in \overline{(C^{n-1} - H^{n-1})}$ and from the induction assumption $f_i^{n-1}(\bar{z}) = d$ which implies $f_i^n(z) = d$. For $z \in H^n$ such that $z_n = b_n^n$ we have $f_i^n(z) = \min(d, \max(f_i^{n-1}(\bar{z}), -d)) = f_i^{n-1}(\bar{z})$, i.e. f_i^n is continuous. For $z \in H^n$ such that $z_n = b_n^n - 2\frac{d}{K}$ we have $f_i^n(z) = \min(d, \max(f_i^{n-1}(\bar{z}), -d)) = f_i^{n-1}(\bar{z})$, i.e. f_i^n is continuous.

Thus all of f_i^n are continuous which implies continuity of f^n . The function f^n satisfies a Lipschitz condition with constant K since it is defined by taking minima and maxima of Lipschitz functions with constant K . Now we show that α^n is the only zero of f^n . Obviously f^n can have zeros only inside H^n . Let $z \in H^n$ be such that:

$$(5.11) \quad z_n \leq b_n^n - \frac{2d}{K}.$$

Then $|z_n - (b_n^n + \frac{d}{K})| \geq 3\frac{d}{K}$ so $\|z - y_i\| \geq 3\frac{d}{K} \forall y_i, i = 1, \dots, 2^{n-2}$. This yields that $g^n(z) = \min(d, d, \dots, d) = d$, thus f^n has no zeros in this domain.

Take

$$(5.12) \quad z_n \geq b_n^n + 2\frac{d}{K}.$$

Then $|z_n - b_n^n| + 2\frac{d}{K}$ which combined with the induction assumption $f_i^{n-1}(\bar{z}) \leq d$ yields $f_i^n(z) = \min(d, \max(f_i^{n-1}(\bar{z}), d)) = d$. Thus f_n has no zeros in this domain.

Take now

$$(5.13) \quad b_n^n - 2\frac{d}{K} \leq z_n \leq b_n^n.$$

In this domain, by the induction assumption the only zeros of $f_j^n, j = 1, \dots, n-1$ are $(\alpha_1^{n-1}, \dots, \alpha_{n-1}^{n-1}, \dots, \alpha_{n-1}^{n-1}, z_n)$. But g^n is zero only for one of these points, namely with $z_n = b_n^n - \frac{d}{K}$. To see this recall that $|\alpha_j^{n-1} - b_j^n| \leq \frac{d}{K}, j = 1, \dots, n-1$ and $\alpha_n^n = b_n^n - \frac{d}{K}$, thus by the definition (5.9) $\|\alpha^n - y_i\| = 2\frac{d}{K}$ for $i = 1, \dots, 2^{n-2}$ so $g^n(\alpha^n) = 0$. For every $z \in H^n$ such that $b_n^n - \frac{d}{K} < z_n < b_n^n$ there exists a y_i with $i_q = 1$ for $\alpha_q^{n-1} \geq b_q^n$ and $i_q = -1$ for $\alpha_q^{n-1} < b_q^n$ such that $\|y_i - z\| < 2\frac{d}{K}$, thus $g^n(z) < 0$. For every $z \in H^n$ such that $b_n^n - 2\frac{d}{K} < z_n \leq b_n^n - \frac{d}{K}$ and for every y_i we have $\|y_i - z\| > 2\frac{d}{K}$, thus $g^n(z) > 0$. Therefore α^n is the only zero of f_n^n in this domain.

For

$$(5.14) \quad b_n^n < z_n \leq b_n^n + 2\frac{d}{K}.$$

We shall take any z such that $g^n(z) = 0$ and show that $\exists i \in \{1, \dots, n-1\}$ such that $f_i^n(z) \neq 0$. Observe first that if $|z_j - b_j^n| \geq 2\frac{d}{K}$ for some $j = 1, \dots, n-1$, then $\|z - b^{n-1}\| \geq 2\frac{d}{K}$ and from the induction assumption (5.7) there exists an i such that $f_i^{n-1}(\bar{z}) = d$, which implies $f_i^n(z) = d$ since $|z_n - b_n^n| \leq 2\frac{d}{K}$. Thus assume that $|z_j - b_j^n| < 2\frac{d}{K}, \forall j = 1, \dots, n-1$, and take z such that $g^n(z) = 0$. This means that

$$(i) \quad \forall j, \quad 1 \leq j \leq 2^{n-2}, \quad \|z - y_j\| \geq 2\frac{d}{K},$$

and

$$(ii) \quad \exists j' : \|z - y_{j'}\| = 2\frac{d}{K}.$$

Suppose that $y_{j'} = y(i_1, \dots, i_{n-1})$ where

$$\left| z_q - b_q^n + i_q \cdot \frac{d}{K} \right| = 2 \frac{d}{K} \text{ for } q \in Q_1,$$

and

$$\left| z_q - b_{q_n} + i_q \cdot \frac{d}{K} \right| < 2 \frac{d}{K}, \text{ for } q \in Q_2,$$

where $Q_1 \neq \emptyset$ and $Q_1 \cup Q_2 = \{1, \dots, j-1, j+1, \dots, n-1\}$. Thus for every $q \in Q_1$, we have $|z_q - b_q^n| = 3 \frac{d}{K}$ or $|z_q - b_q^n| = \frac{d}{K}$. If $\exists q \in Q_1$ such that $|z_q - b_q^n| = 3 \frac{d}{K}$ then $\|z - b^n\| \geq 3 \frac{d}{K}$ and (5.7) implies that $f_i^n(z) = d$ for some i . Otherwise, (i.e. if $|z_q - b_q^n| = \frac{d}{K}$ for all $q \in Q_1$), $z_q = b_q^n \pm \frac{d}{K}$. Then take $y(i_1, \dots, i_{n-2})$ such that i_q are as above for $q \in Q_2$, and for $q \in Q_1$ take $i_q = +1$ if $z_q = b_q^n + \frac{d}{K}$ and $i_q = -1$ if $z_q = b_q^n - \frac{d}{K}$. This implies that $\|y(i_1, \dots, i_{n-2}) - z\| < 2 \frac{d}{K}$ which contradicts (i) and completes the proof of the existence and uniqueness of the zero of f^n .

Obviously $\|\alpha^n - b^n\| = \frac{d}{K}$ since $\alpha^n = b_n^n - \frac{d}{K}$ and $|\alpha_i^n - b_i^n| \leq \frac{d}{K}$ for $i = 1, \dots, n-1$. Also note $\text{dist}(\alpha^n, B^n) = \text{dist}(\alpha^{n-1}, B^{n-1}) = \frac{d}{K}$, thus (5.2) holds.

Equations (5.3) and (5.7) follow immediately from the definition of f^n and the continuity.

Now we show that (5.4) holds. Obviously (5.3) implies (5.4) for $z \in \partial C^n - B^n$. Therefore let $z \in B^n$ and subdivide B^n into 5 regions, $B_i^n, 1, \dots, 5$ (see Figure 5.2 for the case $n=3$), where

$$B_n = \bigcup_{i=1}^5 B_i^n$$

and

$$B_1^n = \left\{ z \in B^n : z_n \geq b_n^n + 2\frac{d}{K} \right\},$$

$$B_2^n = \left\{ z \in B^n : b_n^n \leq z_n < b_n^n + 2\frac{d}{K} \text{ and } \|z - b^n\| \geq 2\frac{d}{K} \right\},$$

$$B_3^n = \left\{ z \in B^n : b_n^n \leq z_n < b_n^n + 2\frac{d}{K} \text{ and } \|z - b^n\| < 2\frac{d}{K} \right\},$$

$$B_4^n = \left\{ z \in B^n : b_n^n - \frac{2d}{K} \leq z_n < b_n^n \right\},$$

$$B_5^n = \left\{ z \in B^n : z_n < b_n^n - 2\frac{d}{K} \right\}.$$

Then recall that (5.6) holds and

- (a) $\forall z \in B_1^n$, by an argument similar to (5.12) we have $f_i^n(z) = d$, $i = 1, \dots, n-1$, thus $\|f^n(z)\| = d$.
- (b) $\forall z \in B_2^n$ the same argument as (5.14) yields $f_i^n(z) = d$ for some $i \in \{1, \dots, n-1\}$.
- (c) $\forall z \in B_3^n$ we have $|z_n - (b_n^n + \frac{d}{K})| \leq \frac{d}{K}$, $|z_i - b_i^n| < 2\frac{d}{K}$ and obviously $z_j = b_j^n = 1$. Let $Q_1 = \{i : z_i \geq b_i^n\}$ and $Q_2 = \{i : z_i < b_i^n\}$. Then for $i \in Q_1$ we have $|z_i - (b_i^n + \frac{d}{K})| \leq \frac{d}{K}$, and for $i \in Q_2$ we have $|z_i - (b_i^n - \frac{d}{K})| \leq \frac{d}{K}$. Thus for $y(i_1, \dots, i_{n-2})$ such that $i_q = 1$ for $q \in Q_1$ and $i_q = -1$ for $q \in Q_2$ we get $\|y(i_1, \dots, i_{n-2}) - z\| \leq \frac{d}{K}$ which yields $\tilde{y}_i(z) = -d$ for some $1 \in \{i, \dots, 2^{n-2}\}$, i.e. $g^n(z) = -d$.
- (d) $\forall z \in B_4^n$ the induction assumption (5.4) and the definition of f^n yield $\|f^{n-1}(\bar{z})\| = d$, therefore $\|f^n(z)\| = d$.
- (e) $\forall z \in B_5^n$ (5.11) implies that $g^n(z) = d$, therefore $\|f^n(z)\| = d$.

Thus $\forall z \in \partial C^n$ we have $\|f^n(z)\| = d$, which completes the proof of (5.4).

For (5.5) note that for z close to α^n by the induction assumption and definition of f_j^n we have $\frac{\partial f_j^n}{\partial z_i}(z) = 0$, $\forall i = 1, \dots, n$. Thus we need to show only $\frac{\partial g^n}{\partial z_i} \Big|_{\alpha^n} = \pm K \cdot \delta_{in}$, $i = 1, \dots, n$. Let $y(i_1, \dots, i_{n-2})$ be such that for $\alpha_q^n \geq b_q^n$ we have $i_q = +1$ and for $\alpha_q^n < b_q^n$ we have $i_q = -1$. Then $|\alpha_q^n - (b_q^n + i_q \cdot \frac{d}{K})| \leq \frac{d}{K}$ since $\|\alpha^n - b^n\| \leq \frac{d}{K}$, and obviously $|\alpha_j^n - b_j^n| \leq \frac{d}{K}$. For z in a small neighborhood of α^n we have

$$\begin{aligned} g^n(z) &= \min_{i=1, \dots, 2^{n-2}} (-2d + K \cdot \|z - y_i\|) \\ &= -2d + K \cdot \min_{i=1, \dots, 2^{n-2}} \|z - y_i\| \\ &= -2d + K \cdot \|z - y(i_1, \dots, i_{n-2})\| \end{aligned}$$

Thus $g^n(z) = -2d + K \cdot |z_n - (b_n^n + \frac{d}{K})|$, and therefore

$$\frac{\partial g^n}{\partial z_i} \Big|_{\alpha^n} = \frac{\partial}{\partial z_i} [K \cdot (-z_n + (b_n^n + \frac{d}{K}))] \Big|_{\alpha^n} = -K \cdot \delta_{in},$$

which shows (5.5).

Now we show (5.7). Observe that (5.3) implies (5.7) for $z \in C^n - H^n$. For $z \in H^n$ such that $z_n \leq b_n^n - 2\frac{d}{K}$ or $z_n \geq b_n^n + 2\frac{d}{K}$ (5.7) follows directly from the proofs following (5.11) and (5.12). For $z \in H^n$ such that $b_n^n - \frac{2d}{K} < z_n \leq b_n^n$ and $\|z - b^n\| \geq 2\frac{d}{K}$ we have $\|\bar{z} - b^{n-1}\| \geq 2\frac{d}{K}$ and then by the induction assumption there exists j such that $f_j^{n-1}(\bar{z}) = d$, so $f_j^n(z) = d$. For $z \in H_n$ such that $b_n^n < z_n \leq b_n^n + 2\frac{d}{K}$ and $\|z - b^n\| \geq 2\frac{d}{K}$ as in (5.14) we have $\|\bar{z} - b^{n-1}\| \geq 2\frac{d}{K}$ and by the induction assumption there exists j such that $f_j^{n-1}(\bar{z}) = d$ which combined with the definition of f^n yields $f_j^n(z) = d$, which completes the proof of (5.7).

The function f^n is in F since it satisfies a Lipschitz condition with constant K and its norm is exactly d on the boundary of C^n (see (5.4)).

This finally completes the proof of Lemma 5.1. (We bet you thought it would never end.) ■

We are now ready to prove Theorem 5.1.

First let $P = \lfloor \frac{K}{8d} \rfloor$ and we show that for every f in F and every sequential (adaptive) information $N_m(f) = [f(x_1), \dots, f(x_m)]$, with $m \leq 2nP^{n-1} - 1$, there exists a cube $H^n \subset C^n$ with $\text{diam}(H^n) = 8\frac{d}{K}$ satisfying (5.1), and such that no point x_i belongs to B^n . Indeed, subdivide the boundary of C^n into $2nP^{n-1}(n-1)$ cubes of diameter $1/P$ by subdividing uniformly each $(n-1)$ face of C^n into $P^{n-1}(n-1)$ -cubes. Then since $m \leq 2nP^{n-1} - 1$ there must exist at least one $(n-1)$ -cube in this subdivision, say \tilde{B}^n , which does not contain any of the x_i points. Since $\text{diam } \tilde{B}^n = 1/P \geq 8\frac{d}{K}$, take as B^n any $(n-1)$ cube of diameter $8\frac{d}{K}$, contained in \tilde{B}^n , with faces parallel to the corresponding faces of \tilde{B}^n . This B^n is obviously an $(n-1)$ face of a cube H^n satisfying (5.1).

Let $f^{**}(z) = [d, \dots, d], \forall z \in C^n$, and let H^n be constructed as above. Let $f^* = f^n$ from Lemma 5.1 applied to this cube H^n . Observe that

$$(5.15) \quad N_m(f^{**}) = N_m(f^*),$$

since for every x_i , $f^{**}(x_i) = f^*(x_i) = [d, \dots, d]$. Moreover there exists a unique zero α^n of f^* . Let D be an open neighborhood of α^n such that f^* is continuously differentiable in D . Then since α^n is a simple zero of f^* , the degree $\deg(f^*, D, \theta) = \pm 1$. Also since f^* has no zeros in $C^n - D$ then $\deg(f^*, C^n - D, \theta) = 0$. Thus by the additivity of degree we get

$$\deg(f^*, C^n, \theta) = \deg(f^*, D, \theta) + \deg(f^*, C^n - D, \theta) = \pm 1.$$

Obviously $\deg(f^{**}, C^n, \theta) = 0$, which combined with (5.15) completes the proof. ■

REFERENCES

- Allgower, E. L., Keller, C. L., Reeves, T. E. (1971), *A Program for the Numerical Approximation of a Fixed Point of an Arbitrary Continuous Mapping of the n-cube or n-cube into itself.*, Aerospace Research Laboratories Tech. Rep. 71-0257, Wright-Patterson Air Force Base, Ohio.
- Boult, T., Sikorski, K. (1985a), *Complexity of Computing Topological Degree of Lipschitz Functions in Two Dimensions*, Tech.Report, Dept. of Computer Science,, Columbia University.
- Boult, T., Sikorski, K. (1985b), *A Fortran Subroutine for Computing Topological Degree of Lipschitz Functions*, in progress.
- Boult, T. (1986), *Information Based Complexity in Nonlinear Equations and Computer Vision*, Ph.D. Thesis in progress..
- Jeppson, M. M. (1972), *A Search for the Fixed Points of a Continuous Mapping*, "Mathematical Topics in Economics Theory and Computation", R. H. Day and S. M. Robinson, Eds., pp. 122-129.
- Kearfott, R. B. (1977), *Computing the Degree of Maps and a Generalized Method of Bisection*, Ph.D. dissertation,, University of Utah, SLC.
- Kearfott, R. B. (1979), *An Efficient Degree-Computation Method for a Generalized Method of Bisection*, Num. Math. 32, 109-127.
- Ortega, J. M., Rheinboldt, W. C. (1970), "Iterative Solution of Nonlinear Equations in Several Variables", Academic Press, NY.
- Sikorski, K. (1982), *Bisection is Optimal*, Num. Math. 40, 111-117.
- Stenger, F. (1975), *Computing the Topological Degree of a Mapping in R^n* , Num. Math. 25, 23-38.
- Stynes, M. (1979a), *An Algorithm for Numerical Calculation of Topological Degree*, Appl. Anal. 9, 63-77.
- Stynes, M. (1979b), *A Simplification of Stenger's Topological Degree Formula*, Num. Math 33, 147-156.
- Stynes, M. (1981), *On the Construction of Sufficient Refinements for Computation of Topological Degree*, Num. Math 37, 453-462.
- Traub, J. F., Wasilkowski, G. W. , Woźniakowski, H. (1983), "Information, Uncertainty and Complexity", Addison Wesley, Reading, Mass..