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What is the Complexity of the Fredholm Problem of the Second Kind?

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ABSTRACT. This paper deals with the approximate solution of the Fredholm problem $Lu = f$ of the second kind, with $f \in W^{r,p}(I)$. Of particular interest is the quality of the finite element method (FEM) of degree k using n inner products of f. The error of the approximation is measured in the $L_p(I)$-norm. We find that the FEM has minimal error iff $k \geq r - 1$. However in the Hilbert case $p = 2$, there always exists a linear combination (called the spline algorithm) of the inner products used by the FEM which *does* have minimal error; this holds regardless of whether $k \geq r - 1$. We also investigate the case where the inner products used by the FEM are not available. Suppose, however, that we can evaluate $f(x)$ for any $x \in I$. In this case, it is reasonable to consider a finite element method with quadrature (FEMQ), in which the inner products required by the FEM are approximated via numerical quadrature. We prove that the FEMQ has minimal error iff $k \geq r - 1$. Moreover, we show that the asymptotic penalty for using the FEM or FEMQ with a value of k that is too small is unbounded.

1. INTRODUCTION

This paper is a theoretical study of the approximate solution of the Fredholm integral equation $Lu = f$ of the second kind, where $f \in W^{r,p}(I)$ and I is the unit interval (see Section 2). There is a vast literature dealing with the numerical solution of these problems. See, e.g., the books [**1**], [**2**], [**4**], [**6**], [**11**], [**17**] and the review article [**12**], as well as the references cited there.

We want to approximate the solution of u of the problem $Lu = f$ by using the values of n linear functionals of f, typical examples being

- n inner products of f, and
- the values of f at n points in I.

We address two problems:

(i) Given the values of n linear functionals of f, how may they be combined so as to approximate the solution of the Fredholm problem with the smallest possible error?
(ii) What is the best set of n linear functionals to use?

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This problem was also studied in [8]. Our treatment generalizes that of [8] in two ways. First, we allow $f \in W^{r,p}(I)$ for $1 < p \leq \infty$, whereas in [8], it is assumed that $f \in C^r(I)$. Second, we allow *any* linear functionals of f, whereas in [8], only function evaluations are considered. For the sake of exposition, we only consider the case where r is a non-negative integer in this paper. However, our results and proofs may be extended via interpolation theory [**21**] to the case of arbitary $r \geq 0$, provided that when r is not an integer, we take $p < \infty$.

In particular, we are interested in the quality of the finite element method (FEM) of degree k (as defined in Section 3) which uses n inner products $(f, s_1), \ldots, (f, s_n)$. Here, $\{ s_1, \ldots s_n \}$ is a basis for a finite element subspace consisting of piecewise polynomials of degree k and $(\cdot, \cdot)$ is the $L_2(I)$ inner product. Note that the FEM does not use the values of r and p.

We show that the FEM has minimal error (to within a constant factor) iff $k \geq r - 1$. Moreover, this minimal-error property of the FEM holds for *any* value of $p \in (1, \infty]$ and for any $r \leq k + 1$. This is important for the following reason. In practice, we generally do not know the exact smoothness of f, i.e., we do not know the best values of r and p such that $f \in W^{r,p}(I)$. So, it would be useful to have an algorithm which is optimal for a large range of r and p. If one agrees that r is restricted to $r \leq k + 1$, the FEM is such a method. Of course, it would be more interesting to find methods which have minimal error for *all* r. We do not know of any methods for the Fredholm problem for which this holds.

We next ask why the error of the FEM is not minimal (to within a constant) when $k < r - 1$. Is it due to the fact that the n inner products used by the FEM are inherently bad, or is it because the FEM uses these inner products inefficiently when $k < r - 1$? We give a partial answer to this question, by restricting our attention to the Hilbert case $p = 2$. For *any* values of k and r, there is a linear combination of the inner products used by the FEM which yields smallest error among all methods using these same inner products. This linear combination is called the spline algorithm. In Section 3, we show that the error of this spline algorithm is minimal (to within a constant factor), no matter what the values of k and r happen to be. This means that the inner products used by the FEM form a best set of linear functionals. Hence in the Hilbert case, the reason that the FEM is not optimal when $k < r - 1$ is that it does not make good use of its information.

When approximating the solution u of the Fredholm problem $Lu = f$, the "pure" FEM (as defined in Section 3) requires certain inner products (i.e., the exact values of the integrals of f multiplied by each of the basis functions for the finite element space). Often, these inner products are not available. Suppose, however, that we can evaluate $f(x)$ for any $x \in I$. In this case, the integrals that appear in the definition of the FEM may be approximated by numerical quadratures (i.e., weighted sums of the f, sampled at various points in the interval). In Section 4, we examine such a finite element method with quadrature (FEMQ) using n evaluations of f. We show that the error of the FEMQ is minimal (to within a constant) iff $k \geq r - 1$. Hence, the values of f at n points in I form a best set of linear functionals. Once again, the results in this section are independent of the value of $r \leq k + 1$ and $p \in (1, \infty]$, so that the same method is optimal for a wide range of r and p.

Finally in Section 5, we discuss the complexity (i.e., the minimal cost) of obtaining approximate solutions whose error is less than ϵ. We find that the FEM and FEMQ produce approximations with minimal cost (to within a constant factor) iff $k \geq r-1$. Moreover, we show that the asymptotic (as $\epsilon \to 0$) complexity penalty for using the FEM or FEMQ with a value of k that is too small is unbounded.

2. THE FREDHOLM PROBLEM OF THE SECOND KIND

In this section, we define the problem to be studied, and prove a regularity theorem which allows estimation of (derivatives of) the solution in terms of (derivatives of) the data. We use standard notation for Sobolev spaces, norms, etc., as found in [5].

Let I denote the unit interval $[0,1]$, let r be a non-negative integer, and let $p \in (1, \infty]$. We are given a function $k: I \times I \to \mathbb{R}$ such that $\partial_1^j k$ is continuous for $0 \leq j \leq r$, where ∂_i^j denotes the jth partial derivative with the ith variable. Define a linear operator $K: L_p(I) \to L_p(I)$ by

$$(Kv)(x) = \int_I k(x,y) v(y)\, dy.$$

Then K is compact. (See [**7**, pg. 518] for a number of alternative conditions which yield compactness of K.) We also assume that 1 is not an eigenvalue of K. Set

$$L = I - K.$$

Then L is an invertible bounded linear operator on $L_p(I)$. Hence, L has a bounded inverse on $L_p(I)$.

We now describe the problem to be studied. Define the linear operator

$$S: W^{r,p}(I) \to L_p(I)$$

by letting

$$u = Sf \text{ iff } Lu = f.$$

By the remarks above, we see that S is a bounded linear transformation, which is an isomorphism when $r = 0$, and (by the Rellich-Kondrasov theorem [5, pg. 118]) is compact when $r > 0$.

We next state and prove a useful regularity theorem.

THEOREM 2.1. *There exists $\alpha = \alpha_r \geq 1$ such that*

$$\alpha^{-1} \|Sf\|_{r,p} \leq \|f\|_{r,p} \leq \alpha \|Sf\|_{r,p} \qquad \forall f \in W^{r,p}(I).$$

PROOF: When $r = 0$, this holds with $\alpha_0 = \max\{ \|S\|, \|L\| \}$. We now suppose that r is a positive integer. Let $u \in W^{r,p}(I)$. Set

$$\beta_j(x) = \left(\int_I |\partial_1^j k(x,y)|^{p'}\, dy \right)^{1/p'},$$

where p' is the exponent conjugate to p, i.e.,

$$p' = \frac{p}{p-1}.$$

We then have

$$|(Ku)^{(j)}(x)| = \left| \int_I \partial_1^j k(x,y)v(y)\,dy \right| \le \beta_j(x)\|u\|_{0,p}.$$

Hence

$$\|Ku\|_{r,p} \le \gamma_r \|u\|_{0,p} \le \alpha_0\gamma_r\|Lu\|_{0,p} \le \alpha_0\gamma_r\|Lu\|_{r,p},$$

where

$$\gamma_r = \left(\sum_{j=0}^{r} \|\beta_j\|_{0,p}^p\right)^{1/p}.$$

Hence, letting $\alpha_r = 1 + \alpha_0\gamma_r$, we have

$$\|u\|_{r,p} = \|(L+K)u\|_{r,p} \le \|Lu\|_{r,p} + \|Ku\|_{r,p} \le \alpha_r\|Lu\|_{r,p},$$

On the other hand, $\alpha_0 \ge 1$ yields

$$\|Lu\|_{r,p} \le \|u\|_{r,p} + \|Ku\|_{r,p} \le \|u\|_{r,p} + \gamma_r\|u\|_{0,p} \le (1+\gamma_r)\|u\|_{r,p} \le \alpha_r\|u\|_{r,p}.$$

Thus

$$\alpha_r^{-1}\|u\|_{r,p} \le \|Lu\|_{r,p} \le \alpha_r\|u\|_{r,p} \qquad \forall u \in W^{r,p}(I).$$

The result desired follows immediately from this inequality, the bounded inverse theorem, and the Fredholm alternative theorem.

3. Minimal error properties of the finite element method.

In this section, we discuss the finite element method (FEM) for the Fredholm problem. First, we define the FEM of degree k. We then explain the concept of a minimal-error algorithm, and give a tight estimate of the nth minimal error for our problem. Next, we give a sharp estimate of the error of the FEM. It then follows that the error of the FEM is minimal (to within a constant) iff

$$k \ge r - 1, \tag{3.1}$$

k being the degree of the finite element subspace. We also consider the situation where (3.1) does not hold.

We first define the FEM, using the notation of [**5**]. Let k be a non-negative integer. For a non-negative integer n, we let S_n be an n-dimensional space of piecewise polynomials of degree k, with no inter-element continuity imposed. That is, let

$$\Delta_n = \{\, 0 = \xi_0 < \xi_1 < \cdots < \xi_m = 1 \,\} \qquad \text{and} \qquad m(k+1) = n$$

denote a grid on I. (In the remainder of this paper, we assume that the grid sequence $\{\,\Delta_n\,\}_{n=1}^{\infty}$ is *quasi-uniform* [**16**, pg. 272].) Let

$$I_l = [\xi_{l-1}, \xi_l] \qquad (1 \le l \le m)$$

denote the lth subinterval in the grid Δ_n. Then

$$s \in S_n \qquad \text{iff} \qquad s|_{I_l} \in P_k(I_l) \qquad (1 \le l \le m).$$

Note that $S_n \subset L_q(I)$ for any $q \in [1, \infty]$.

The *finite element method* (FEM) is then defined as follows. Let $f \in W^{r,p}(I)$. For each non-negative integer n, an approximation $u_n \in S_n$ to $u = Sf$ is chosen such that

$$B(u_n, s) = (f, s) \qquad \forall s \in S_n. \tag{3.2}$$

The bilinear form $B: L_p(I) \times L_{p'}(I) \to \mathbb{R}$ in (3.2) is defined by

$$B(u, v) = ((I - K)u, v) \qquad \forall u \in L_p(I), v \in L_{p'}(I).$$

Here, the duality pairing of $L_p(I)$ and $L_{p'}(I)$ is denoted $(\cdot, \cdot)$, i.e.,

$$(g, v) = \int_I g(x)v(x)\, dx \qquad \forall g \in L_p(I), v \in L_{p'}(I),$$

where (as always) p' denotes the exponent conjugate to p. Of course when $p = 2$, this means that $(\cdot, \cdot)$ is the usual inner product on $L_2(I)$.

To make the specification of the FEM more precise, we let $\{ s_1, \ldots, s_n \}$ be a basis for S_n. Consider the *Gram matrix*

$$G = \left[B(s_j, s_i)\right]_{1 \le i,j \le n}.$$

Later on, we will establish that G is invertible for n sufficiently large. Let

$$\alpha = G^{-1}\beta \qquad \text{with} \qquad \beta_i = (f, s_i) \qquad (1 \le i \le n).$$

We have

$$u_n = \sum_{j=1}^{n} \alpha_j s_j.$$

Examining this expression, we come to an important conclusion. *The approximation produced by the FEM depends on f only through the inner products of f with the basis functions of the finite element subspace S_n*, i.e., through

$$N_n f = \begin{bmatrix} (f, s_1) \\ \vdots \\ (f, s_n) \end{bmatrix}.$$

We indicate this explicitly by writing

$$u_n = \varphi_n(N_n f),$$

where (as before) u_n denotes the approximation produced by the FEM. We refer to N_n as *finite element information* (FEI), since $N_n f$ is the only knowledge the FEM has of a right-hand side f.

We measure the quality of the approximations produced by the FEM in the $L_p(I)$ norm. That is, we define the *error* of the FEM φ_n to be

$$e(\varphi_n) = \sup_{f \in F} \|Sf - \varphi_n(N_n f)\|_{0,p}, \tag{3.3}$$

where F denotes the unit ball of $W^{r,p}(I)$

$$F = BW^{r,p}(I) := \{\, f \in W^{r,p}(I) : \|f\|_{r,p} \le 1 \,\}.$$

Since φ_n is a homogeneous function, the error $e(\varphi_n)$ also satisfies

$$e(\varphi_n) = \sup \left\{ \frac{\|Sf - \varphi_n(N_n f)\|_{0,p}}{\|f\|_{r,p}} : f \in W^{r,p}(I), f \neq 0 \right\}. \tag{3.4}$$

Of course, the FEM is not the only method using the FEI. It is natural to ask whether there are any better methods using this information. In other words, is there a better combination (not necessarily linear) of the inner products making up the FEI whose error is better than that of the FEM? Let

$$e(N_n) = \inf_{\varphi} \sup_{f \in F} \|Sf - \varphi(N_n f)\|_{0,p}.$$

Here, φ is any mapping (possibly nonlinear) which uses the inner products in $N_n f$ to approximate Sf. We say that φ is an *algorithm* using N_n. Thus $e(N_n)$ measures the minimal error among all algorithms using N_n.

We finally ask whether there is any better set of inner products to use, so as to minimze the error. Let

$$e(n) = \inf_{N} e(N),$$

the infimum being over all information N consisting of n linear functionals. We say that $e(n)$ is the nth *minimal error*. An algorithm using n linear functionals whose error equals $e(n)$ is said to be an nth *minimal error algorithm*.

It is desirable to find nth minimal error algorithms for each n. In this paper, we are content to pursue a more modest goal. We seek a sequence of algorithms, each using n linear functionals, such that the error of the nth algorithm in the sequence is at most a constant multiple of the nth minimal error. Adopting the terminology of [**22**], we shall refer to such a sequence as being a *quasi-minimal* sequence of algorithms. It is of particular interest to find conditions which are necessary and sufficient for the FEM to be quasi-minimal.

To do this, we must first establish tight bounds on the nth minimal error. This will give a benchmark, against which we may measure the error of the FEM. The result is expressed in the big-theta notation of [**15**]. That is, we say that

$$f = \Theta(g) \qquad \text{iff} \qquad f = O(g) \text{ and } f = \Omega(g),$$

where

$$f = \Omega(g) \qquad \text{iff} \qquad g = O(f).$$

THEOREM 3.1. *The nth minimal error satisfies*

$$e(n) = \Theta(n^{-r}) \text{ as } n \to \infty.$$

PROOF: When $p < \infty$, we may use [**20**, Theorem 2.6.1], Theorem 2.1, and [**21**, Theorem 4.10.2] to see that

$$e(n) = \Theta\Big(d^n(SF, L_p(I))\Big) = \Theta\Big(d^n(F, L_p(I))\Big) = \Theta(n^{-r}).$$

Here, d^n denotes the Gelfand n-width. This establishes the result for the case $p < \infty$.

We now turn to the case $p = \infty$. By [**20**, Corollary 3.3.1 and Theorem 3.5.1], the fact that the linear Kolmogorov n-width λ_n dominates the Kolmogorov n-width d_n, and Theorem 2.1, we have

$$e(n) = \lambda_n(SF, L_\infty(I)) \geq d_n(SF, L_\infty(I)) = \Theta\Big(d_n(F, L_\infty(I))\Big).$$

Let

$$F_0 = \{\, f \in W_0^{r,\infty}(I) : \|f^{(r)}\|_{0,\infty} \leq 1 \,\}.$$

The Poincaré's inequality yields

$$d_n(F, L_\infty(I)) = \Theta(d_n(F_0, L_\infty(I)).$$

Since the proof of [**19**, Theorem 2.76] actually establishes that

$$d_n(F_0, L_\infty(I))) = \Theta(n^{-r}),$$

we have the lower bound

$$e(n) = \Omega(n^{-r}).$$

It remains to show that

$$e(n) = O(n^{-r}).$$

Since $F \subseteq F_0$, we have

$$\lambda_n(F, L_\infty(I)) \leq \lambda_n(F_0, L_\infty(I)) = \Theta(n^{-r})$$

see [**13**, pg. 182]. Hence Theorem 2.1 yields

$$e(n) = \lambda_n(SF, L_\infty(I)) = \Theta\Big(\lambda_n(F, L_\infty(I))\Big) = O(n^{-r}).$$

Thus we find that $e(n) = \Theta(n^{-r})$ when $p = \infty$.

We now determine the $L_p(I)$-error of the FEM, using the results in [**12**]. Let $P_n\colon L_2(I) \to L_2(I)$ be the orthogonal projector of $L_2(I)$ onto S_n, i.e., for any $h \in L_2(I)$, $P_n h \in S_n$ satisfies

$$(P_n h, s) = (h, s) \qquad \forall s \in S_n. \tag{3.5}$$

(This makes sense because $S_n \subset L_2(I)$.)

LEMMA 3.1. *For any $q \in [1,\infty]$, there exists $\pi_q > 0$ such that for any non-negative integer n,*

$$\|P_n v\|_{0,q} \le \pi_q \|v\|_{0,q} \qquad \forall v \in L_q(I),$$

and so

$$\|v - P_n v\|_{0,q} \le (1+\pi_q) \inf_{s\in S_n} \|v - s\|_{0,q}.$$

PROOF: In order to establish the first inequality, let $v \in L_q(I)$. Let I_l be a subinterval in Δ_n. By [**5**, (3.2.33)], there exists $C > 0$, independent of v and n, such that

$$\|P_n v\|_{0,q,I_l} \le C n^{1/2-1/q} \|P_n v\|_{0,2,I_l}. \tag{3.6}$$

Let

$$w_n = \begin{cases} P_n v & \text{on } I_l, \\ 0 & \text{otherwise.} \end{cases}$$

Using the facts that $P_n v = w_n$ on I_l, $w_n = 0$ outside of I_l, P_n is self-adjoint, and $P_n w_n = w_n$ (which holds because $w_n \in S_n$), we find that

$$\|P_n v\|_{0,2,I_l}^2 = (P_n v, w_n)_{I_l} = (P_n v, w_n) = (v, P_n w_n) = (v, w_n) = (v, w_n)_{I_l} = (v, P_n v)_{I_l}.$$

Letting q' denote the exponent conjugate to q, Hölder's inequality yields

$$\|P_n v\|_{0,2,I_l}^2 \le \|v\|_{0,q,I_l} \|P_n v\|_{0,q',I_l}.$$

Once again, [**5**, (3.2.33)] yields the existence of $C > 0$, independent of v and n, such that

$$\|P_n v\|_{0,q',I_l} \le C n^{1/2-1/q'} \|P_n v\|_{0,2,I_l}.$$

Using this inequality with (3.6) and the fact that $1/q + 1/q' = 1$, we have

$$\|P_n v\|_{0,q,I_l} \le C \|v\|_{0,q,I_l}.$$

The desired result now easily follows from this inequality and the discrete version of Hölder's inequality.

To prove the remainder of the lemma, let $v \in L_q(I)$. For any non-negative integer n and $s \in S_n$, we have $P_n s = s$, so that

$$\|v - P_n v\|_{0,q} \le \|v - s\|_{0,q} + \|P_n(s - v)\|_{0,q} \le (1+\pi_q)\|v - s\|_{0,q}.$$

Since $s \in S_n$ is arbitrary, we may take the infimum over all such s to establish the desired inequality, completing the proof of the lemma.

Thus P_n satisfying (3.5) is a bounded linear operator on $L_p(I)$.

We briefly recall the standard approximation-theoretic results concerning S_n, see e.g. [**3**] or [**16**]. Let $s \ge 0$ and $q \in [1,\infty]$. There is a positive constant C such that for any $v \in W^{s,q}(I)$ and any integer n, one can find $v_n \in S_n$ for which

$$\|v - v_n\|_{0,q} \le C n^{-\lambda} \|v\|_{s,q}, \tag{3.7}$$

where

$$\lambda = \min(k+1, s). \tag{3.8}$$

We then have

LEMMA 3.2. *Let $s \geq 0$ and $q \in [1, \infty]$. There is a positive constant C such that for any $v \in W^{s,q}(I)$ and any non-negative integer n,*

$$\|v - P_n v\|_{0,q} \leq C n^{-\lambda} \|v\|_{s,q},$$

with λ given by (3.8). Hence, for any $v \in L_p(I)$,

$$\lim_{n \to \infty} \|v - P_n v\|_{0,q} = 0.$$

PROOF: The first part of the lemma follows immediately from Lemma 3.1 and (3.7). To see the second part, let $v \in L_q(I)$. Given $\epsilon > 0$, choose $v_\epsilon \in C^1(I)$ such that

$$\|v - v_\epsilon\|_{0,q} < \frac{\epsilon}{2(1 + \pi_q)}.$$

Set $n_0(\epsilon) = \lceil 2C\|v_\epsilon\|_{1,q}/\epsilon \rceil$, where C is as in the estimate of the first part of the lemma. Then for any $n \geq n_0(\epsilon)$, we have

$$\|v_\epsilon - P_n v_\epsilon\|_{0,q} \leq C n^{-1} \|v_\epsilon\|_{1,q} \leq \frac{\epsilon}{2}.$$

Moreover, Lemma 4.1 yields

$$\|(v - v_\epsilon) - P_n(v - v_\epsilon)\|_{0,q} \leq (1 + \pi_q)\|v - v_\epsilon\|_{0,q} < \frac{\epsilon}{2}.$$

Hence for any $n > n_0(\epsilon)$,

$$\|v - P_n v\|_{0,q} \leq \|(v - v_\epsilon) - P_n(v - v_\epsilon)\|_{0,q} + \|v_\epsilon - P_n v_\epsilon\|_{0,q} < \epsilon,$$

completing the proof of the lemma.

We are now able to establish that the FEM φ_n is well-defined and uniformly stable for n sufficiently large. In particular, this implies that the Gram matrix G defined previously is invertible for sufficiently large n. We give sharp bounds on the error of the FEM, showing that the FEM is quasi-minimal iff $k \geq r - 1$.

THEOREM 3.2. *There exists a positive integer n_0 such that the FEM is defined for all $n \geq n_0$, as well as a constant C, independent of n, such that*

$$\|\varphi_n(N_n f)\|_{0,p} \leq C\|f\|_{0,p} \qquad \forall f \in W^{r,p}(I).$$

Moreover,

$$e(\varphi_n) = \Theta(n^{-\mu}) \qquad \text{as} \qquad n \to \infty,$$

where

$$\mu = \min(\widetilde{k + 1}, r),$$

so that the FEM is quasi-minimal iff (3.1) holds.

PROOF: Using the formulation of [**12**], we see that $u_n = \varphi_n(N_n f)$ is the solution of

$$(I - P_n K)u_n = P_n f,$$

where the right-hand side is well-defined by Lemma 3.1. Since 1 is not an eigenvalue of K, $L = I - K$ has a bounded inverse on $L_p(I)$. Hence, [**12**, Theorem 2.1] implies that there exists an integer n_0 such that $I - P_n K$ is invertible for all $n \geq n_0$. Moreover, there exists a positive constant C such that

$$\|(I - P_n K)^{-1}\| \leq C$$

and

$$\|u - u_n\|_{0,p} \leq C\|u - P_n u\|_{0,p}$$

for all $n \geq n_0$. These results, with Lemmas 3.1 and 3.2, establish the well-definedness and uniform stability of the FEM, as well as an upper bound $e(\varphi_n) = O(n^{-\mu})$ on the error. To establish a lower bound on the error of the FEM, we may use the techniques of [**23**, Theorem 5.2] to see that there exists a nonzero function $v \in W^{r,p}(I)$ and a constant $C > 0$, such that for all n sufficiently large,

$$\inf_{s \in S_n} \|v - s\|_{0,p} \geq C n^{-\mu} \|v\|_{r,p}. \tag{3.9}$$

Since $v \neq 0$, $Lv \neq 0$. Let $f = Lv/\|Lv\|_{r,p} \in BW^{r,p}(I)$. Then the linearity of S, φ_n, and N_n yield

$$e(\varphi_n) \geq \|Sf - \varphi_n(N_n f)\|_{0,p} = \frac{1}{\|Lv\|_{r,p}}\|v - \varphi_n(N_n Lv)\|_{0,p} \geq \frac{1}{\|Lv\|_{r,p}} \inf_{s \in S_n} \|v - s\|_{0,p}$$

(since $\varphi_n(N_n Lv) \in S_n$). Using (3.9), we thus have

$$e(\varphi_n) \geq \frac{C}{\|Lv\|_{r,p}} n^{-\mu},$$

establishing that $e(\varphi_n) = \Theta(n^{-\mu})$. The final statement of the theorem now follows from this estimate and Theorem 3.1.

Hence the FEM is quasi-minimal iff $k \geq r - 1$. Suppose this inequality no longer holds. We show that in the Hilbert case $p = 2$, the non-optimality of the FEM is due to the fact that it uses its information in a non-optimal manner. To be more precise, let φ_n^s denote the *spline algorithm* using the finite element information N_n [**20**, Chapter 4]. The spline algorithm is a linear combination of the functionals which make up N_n, i.e., there exist elements $u_1^s, \ldots, u_n^s$ of $L_p(I)$ such that

$$\varphi_n^s(N_n f) = \sum_{i=1}^{n} (f, s_i)\, u_i^s.$$

(In fact, in the case where $s_1, \ldots, s_n$ are $H^r(I)$-orthonormal, u_i^s is the exact solution of the problem $Lu_i^s = s_i$.) Moreover, the spline algorithm has the smallest error among *all* algorithms using N_n.

THEOREM 3.3. *For the Hilbert case* $p = 2$,

$$e(\varphi_n^s) = e(N_n) = \Theta(n^{-r}) \qquad \text{as } n \to \infty.$$

PROOF: Since the first equality follows by optimality of the spline algorithm in the Hilbert case [**20**, Theorem 4.5.1], we need only show that the second holds. In order to do this, note that Theorem 3.1 yields

$$e(N_n) \geq e(n) = \Theta(n^{-r}),$$

establishing the lower bound $e(N_n) = \Omega(n^{-r})$. We need only show the upper bound $e(N_n) = O(n^{-r})$, which will be done by using the formula

$$e(N_n) = \sup_{z \in F \cap \ker N_n} \|Sz\|_{0,2};$$

see [**20**, Theorem 3.4.2]. Let $z \in F \cap \ker N_n$, so that

$$(z, s) = 0 \qquad \forall s \in S_n$$

and

$$\|z\|_{r,2} \leq 1.$$

By [**3**, Theorem 4.1.1], there exists $s \in S_n$ such that

$$\|z - s\|_{-r,2} \leq Cn^{-r}\|z\|_{0,2},$$

the positive constant C being independent of z, s, and n. Using the results above with Theorem 2.1, we find

$$\|Sz\|_{0,2} \leq \alpha\|z\|_{0,2} = \frac{\alpha|(z, z - s)|}{\|z\|_{0,2}} \leq \frac{\alpha\|z\|_{r,2}\|z - s\|_{-r,2}}{\|z\|_{0,2}} \leq Cn^{-r}.$$

Taking the supremum over all $z \in F \cap \ker N_n$, we have $e(N_n) = O(n^{-r})$. Hence, $e(N_n) = \Theta(n^{-r})$ as $n \to \infty$, establishing the theorem.

Hence, regardless of whether (3.1) holds, there *always* exists a linear algorithm using FEI (namely, the spline algorithm) which is quasi-minimal.

REMARK 3.1. It is reasonable to ask whether Theorem 3.3 holds for other values of p. The main problem in extending Theorem 3.3 to the case $p \neq 2$ lies in extending [**3**, Theorem 3.1.1] to this case. Most of the proof of that result seems to hold for the case $p \in (1, \infty)$, assuming that spaces of fractional order (which arise in the proof of that result) are defined via complex interpolation [**21**]. However, the proof of that theorem also depends on the optimality of orthogonal projections in the Hilbert setting. The analogous statement, required to prove the extension of [**3**, Theorem 3.1.1] to the non-Hilbert case, would be the uniform boundedness of the $W^{-s,2}(I)$-orthogonal projection onto S_n in the $W^{-s,q}(I)$-norm, which is an extension of Lemma 3.1 of this paper from the $L_q(I)$-norm to the negative norm of $W^{-s,q}(I)$. It is not clear whether this extension holds.

4. Standard information and the FEM with quadrature.

Recall that in approximating the solution u of the problem $Lu = f$, the "pure" finite element method (3.2) requires the values of the integrals

$$\int_I f(x)s_i(x)\,dx \qquad (1 \le i \le n) \tag{4.1}$$

(where $s_1, \dots, s_n$ are the basis functions for the finite element space S_n of degree k and dimension n). The (exact) values of these integrals are not generally available for arbitrary problem elements f. It is more usual to assume that it is possible to compute the values of a problem element f at any point of I. If this is the case, one may approximate the integrals (4.1) by numerical quadrature rules, i.e., weighted sums of the problem element f evaluated at n points in I.

In addition, there is a second kind of integral appearing in (3.2), namely integrals of the form

$$\int_I \left[s_j(x) - \int_I k(x,y)s_j(y)\,dy\right] s_i(x)\,dx \qquad (1 \le i,j \le n). \tag{4.2}$$

It is possible that for special kinds of kernals k, the integrals of the form (4.2) can be evaluated in closed form (since the basis functions are piecewise polynomials). However, for even moderately-complicated kernels, these exact values may be unavailable or difficult to compute. For this reason, one might wish to approximate the integrals (4.2) by a quadrature rule using n^2 values of the kernel k and n values of $s_1, \dots, s_n$.

In this section, we introduce a "finite element method with quadrature," in which integrals of the form (4.1) and (4.2) are approximated via numerical quadrature rules. We show that the error of this FEM with quadrature is essentially the same as that of the "pure" finite element method. From this, it follows that the FEM with quadrature is quasi-minimal under exactly the same conditions that the FEM is quasi-minimal. That is, the FEM with quadrature is quasi-minimal iff $k \ge r - 1$, where k denotes the degree of the finite element subspace.

REMARK 4.1. Our analysis is similar to that of [**5**, Section 4.1]. That is, we establish and then use a weakly coercive [**16**, Section 7.4] version of the First Strang Lemma [**5**, Theorem 4.1.1], rather than try to directly apply the results of [**12**]. The main reason for not using the latter approach is that the projection operator P_n of (3.5) would have to be replaced by a new projection operator. This new operator involves the evaluation of problem elements at points in I, and hence is not defined over all of $L_p(I)$; as a result, this approach would yield estimates which are not sharp.

As in the previous sections, we will be using the notation of [**5**] for Sobolev spaces, norms, seminorms, etc., our exposition closely following that of [**5**, Section 4.1]. In the remainder of this section, we assume that $r \ge 1$, so that the Sobolev embedding theorem implies that $f(x)$ is well-defined for all $x \in I$ and for all $f \in W^{r,p}(I)$, no matter what value of $p > 1$ is chosen. We also assume that $k \in W^{r,\infty}(I \times I)$.

For the sake of exposition, we restrict our attention to the case where the integrals occurring in the definition of the FEM are replaced by piecewise $(k+1)$-point Gauss quadratures. We let $\hat{Q}$ denote a $(k+1)$-point Gauss quadrature rule over the *reference element* $\hat{I} = [-1, 1]$, so that

$$\int_{-1}^{1} \hat{g}(\hat{x})\, d\hat{x} = \hat{Q}(\hat{g}) + \hat{E}(\hat{g}), \tag{4.3}$$

where

$$\hat{Q}(\hat{g}) := \sum_{j=0}^{k} \hat{\omega}_j \hat{g}(\hat{x}_j)$$

and

$$\hat{E}(\hat{p}) = 0 \qquad \forall \hat{p} \in P_{2k+1}(\hat{I}).$$

Using the notation of the previous section, we see that for $0 \le a \le m-1$, this induces a quadrature rule Q_a on $I_a := [\xi_a, \xi_{a+1}]$ by

$$Q_a(g) = \sum_{j=0}^{k} \omega_{ja} g(x_{ja}),$$

where

$$\omega_{ja} = \frac{\xi_{a+1} - \xi_a}{2} \hat{\omega}_j \qquad \text{and} \qquad x_{ja} = \frac{\xi_{a+1} - \xi_a}{2} (\hat{x}_j + 1) + \xi_a.$$

Hence

$$\int_{I_a} g(x)\, dx = Q_a(g) + E_a(g), \tag{4.4}$$

where

$$E_a(g) = \frac{\xi_{a+1} - \xi_a}{2} \hat{E}(\hat{g}) \tag{4.5}$$

and

$$\hat{g}(\hat{x}) = g\left(\frac{\xi_{a+1} - \xi_a}{2} (\hat{x} + 1) + \xi_a \right).$$

We write the nodes $\{ x_{ja} \}$ in increasing order as $x_1, \ldots, x_n$, with $n = (k+1)m$; ω_i is the weight from $\{ \omega_{ja} \}$ corresponding to the node x_i.

Let n be a positive integer. We define a bilinear form B_n approximating B by

$$B_n(v, w) = \sum_{i=1}^{n} \omega_i \left[v(x_i) - \sum_{j=1}^{n} \omega_j k(x_i, x_j) v(x_j) \right] w(x_i).$$

For $f \in W^{r,p}(I)$, we define a linear functional f_n approximating $(f, \cdot)$ by

$$f_n(w) = \sum_{i=1}^{n} \omega_i f(x_i) w(x_i).$$

The *finite element method with quadrature* (FEMQ) is then defined as follows. Given $f \in W^{r,p}(I)$, for each positive integer n, an approximation $u_n \in S_n$ to $u = Sf$ is chosen such that

$$B_n(u_n, s) = f_n(s) \qquad \forall s \in S_n.$$

We make the specification of the FEMQ more precise. Let $\{ s_1, \dots, s_n \}$ be a basis for S_n. Consider the *Gram matrix*

$$G = \left[B_n(s_j, s_i)\right]_{1 \le i,j \le n}.$$

Later on, we will establish that G is invertible for n sufficiently large. Let

$$\alpha = G^{-1}\beta \qquad \text{with} \qquad \beta_i = f_n(s_i) \qquad (1 \le i \le n).$$

We have

$$u_n = \sum_{j=1}^{n} \alpha_j s_j.$$

As in the previous section, this expression shows that the approximation produced by the FEMQ depends on f only through the *standard information*

$$N_n^q f = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

We indicate this explicitly by writing

$$u_n^q = \varphi_n^q(N_n^q f).$$

In the remainder of this section, we determine the error of the FEMQ, i.e., we find a sharp estimate of

$$e(\varphi_n^q) = \sup_{f \in F} \|Sf - \varphi_n^q(N_n^q f)\|_{0,p}.$$

From this estimate, it will follow that the FEMQ is quasi-minimal precisely when the FEM is quasi-minimal. That is, the FEMQ is quasi-minimal iff $k \ge r - 1$.

This error analysis will be based upon the following weakly-coercive version of the First Strang Lemma [5, Theorem 4.1.1]. Here, and in the remainder of the paper, we follow the custom of letting C denote a positive constant (not necesarily the same at each occurrence) which is independent of n and the various functions involved.

LEMMA 4.1. *Suppose that the approximating bilinear forms* $\{ B_n \}$ *are* uniformly weakly coercive, *i.e., there exists a positive integer* n_1 *and a positive constant* β *such that for all* $n \ge n_1$ *and for any* $v \in S_n$*, there is a nonzero* $s \in S_n$ *such that*

$$|B_n(v, s)| \ge \beta \|v\|_{0,p} \|s\|_{0,p'}.$$

Then the FEMQ φ_n^q *is defined for all* $n \geq n_1$. *Moreover, there is a positive constant* C *such that for any* $n \geq n_1$,

$$e(\varphi_n^q) \leq C \sup_{f \in F} \left\{ \inf_{s \in S_n} \left[\|Sf - s\|_{0,p} + \sup_{\substack{w \in S_n \\ w \neq 0}} \frac{|B(s,w) - B_n(s,w)|}{\|w\|_{0,p'}} \right] + \sup_{\substack{w \in S_n \\ w \neq 0}} \frac{|(f,w) - f_n(w)|}{\|w\|_{0,p'}} \right\}.$$

PROOF: We first show that the FEMQ is defined for all $n \geq n_1$. That is, we need to show that the Gram matrix G is invertible, i.e., G has a trivial kernel. Suppose in fact that there exists a nonzero $\varsigma \in \mathbb{R}^n$ such that $G\varsigma = 0$. Letting $z = \sum_{j=1}^n \varsigma_j s_j$, we see that

$$B_n(z,s) = 0 \qquad \forall s \in S_n.$$

Since ς is nonzero and $s_1, \ldots, s_n$ is a basis, z is nonzero. Hence by uniform weak coercivity, there exists nonzero $s \in S_n$ such that

$$|B_n(z,s)| \geq \beta \|z\|_{0,p} \|s\|_{0,p'} > 0,$$

a contradiction. Hence G has a trivial kernel, and the FEMQ is defined for all $n \geq n_1$.

We now establish the error bound. Let $f \in F$, $n \geq n_1$, and $u_n = \varphi_n^q(N_n^q f)$. Let $s \in S_n$. By uniform weak coercivity, there is a nonzero $w \in S_n$ such that

$$\begin{aligned} \beta \|u_n - s\|_{0,p} \|w\|_{0,p'} &\leq |B_n(u_n - s, w)| \\ &\leq |B(Sf - s, w)| + |B(s,w) - B_n(s,w)| + |B_n(u_n, w) - B(Sf, w)| \\ &\leq \alpha_0 \|Sf - s\|_{0,p} \|w\|_{0,p'} + |B(s,w) - B_n(s,w)| + |f_n(w) - f(w)|, \end{aligned}$$

where α_0 is given by Theorem 2.1. Since $w \neq 0$, the above may be divided by $\beta \|w\|_{0,p'}$. Hence, for any $s \in S_n$, there exists nonzero $w \in S_n$ such that

$$\|u_n - s\|_{0,p} \leq \frac{\alpha_0}{\beta} \left[\|Sf - s\|_{0,p} + \frac{|B(s,w) - B_n(s,w)|}{\|w\|_{0,p'}} + \frac{|(f,w) - f_n(w)|}{\|w\|_{0,p'}} \right].$$

Since $u_n = \varphi_n^q(N_n^q f)$, we have

$$\|Sf - \varphi_n^q(N_n^q f)\|_{0,p} \leq \|Sf - s\|_{0,p} + \|u_n - s\|_{0,p}.$$

Letting $C = 1 + \alpha_0/\beta$, the previous two inequalities yield

$$\|Sf - \varphi_n^q(N_n^q f)\|_{0,p} \leq C \left[\|Sf - s\|_{0,p} + \frac{|B(s,w) - B_n(s,w)|}{\|w\|_{0,p'}} + \frac{|(f,w) - f_n(w)|}{\|w\|_{0,p'}} \right].$$

Taking the supremum over nonzero $w \in S_n$, the infimum over $s \in S_n$, and finally the supremum over $f \in F$, we have the desired result.

In order to use this estimate, we will first establish the proper error estimates over each subinterval in the grid, which will then be combined to yield an estimate of the error over the entire interval. To do this, define

$$E_{ab}(v,w) = \int_{I_a}\int_{I_b} k(x,y)v(y)w(x)\,dy\,dx - \sum_{i=0}^{k}\omega_{ia}\left[\sum_{j=0}^{k}\omega_{jb}k(x_{ia},x_{jb})v(x_{jb})\right]w(x_{ia})$$

$$(1 \le a, b \le m).$$

Recalling the definition (4.4) of the error functional E_a, we have the following estimate of the "local consistency error:"

LEMMA 4.2. *Let $\sigma = \min(k+2, r)$. There is a positive constant C, such that for any positive integer n, any v, $w \in S_n$, any $f \in W^{r,p}(I)$, and any a, $b \in \{1,\dots,m\}$, we have the estimates*

$$|E_{ab}(v,w)| \le Cn^{-3}|k|_{1,\infty,I_a\times I_b}\|v\|_{0,p,I_b}\|w\|_{0,p',I_a}\,, \tag{4.6}$$

$$|E_{ab}(v,w)| \le Cn^{-(\sigma+1)}\|k\|_{r,\infty,I_a\times I_b}\|v\|_{r,p,I_b}\|w\|_{0,p',I_a}\,, \tag{4.7}$$

and

$$|E_a(fw)| \le Cn^{-\sigma}\|f\|_{r,p,I_a}\|w\|_{0,p',I_a}\,. \tag{4.8}$$

PROOF: For $c \in \{1,\dots,m\}$, define $\alpha_c = (\xi_c - \xi_{c-1})/2$ and $\beta_c = (\xi_{c-1}+\xi_c)/2$. Then $F_c\colon \hat I \to I_c$, defined by

$$F_c(\hat x) = \alpha_c\hat x + \beta_c,$$

is an affine bijection of $\hat I$ onto I_c. Letting

$$B_{ab} = \begin{bmatrix}\alpha_a & 0\\ 0 & \alpha_b\end{bmatrix} \qquad \text{and} \qquad b_{ab} = \begin{bmatrix}\beta_a\\ \beta_b\end{bmatrix},$$

we define an affine bijection $F_{ab}\colon \hat I \times \hat I \to I_a \times I_b$ by

$$F_{ab}\left(\begin{bmatrix}\hat x\\ \hat y\end{bmatrix}\right) := \begin{bmatrix}F_a\hat x\\ F_b\hat y\end{bmatrix} = B_{ab}\begin{bmatrix}\hat x\\ \hat y\end{bmatrix} + b_{ab}.$$

For any $\hat z\colon \hat I \times \hat I \to \mathbb{R}$, let

$$\hat E_{\hat I\times\hat I}(\hat z) = \int_{\hat I}\int_{\hat I}\hat z(\hat x,\hat y)\,d\hat y\,d\hat x - \sum_{i=0}^{k}\hat\omega_i\sum_{j=0}^{k}\hat\omega_j\hat z(\hat x_i,\hat x_j)$$

denote the error in the tensor $(k+1)\times(k+1)$-point Gauss quadrature rule on $\hat I \times \hat I$. Given a function $z\colon I_a \times I_b \to \mathbb{R}$, we define a function $\hat z\colon \hat I \times \hat I \to \mathbb{R}$ by the change of variables

$$\hat z(\hat x,\hat y) = z(x,y) \qquad \text{where} \qquad x = F_a\hat x\,, \quad y = F_b\hat y\,.$$

Letting v and w be as in the statement of the lemma, we then may use the quasi-uniformity of the family of finite-element spaces to see that

$$(4.9)\qquad |E_{ab}(v,w)| = |\det B_{ab}|\,|\hat{E}_{\hat{I}\times\hat{I}}(\hat{k}\hat{v}\hat{w})| = \alpha_a\alpha_b|\hat{E}_{\hat{I}\times\hat{I}}(\hat{k}\hat{v}\hat{w})| \le Cn^{-2}|\hat{E}_{\hat{I}\times\hat{I}}(\hat{k}\hat{v}\hat{w})|,$$

where C is a fixed positive constant. Analogously, by using (4.3), (4.4), and (4.5), we see that there is a positive constant C such that if f and w are as in the statement of the lemma, then

$$(4.10)\qquad |E_a(fw)| = |\det B_a|\,|\hat{E}(\hat{f}\hat{w})| = \alpha_a|\hat{E}(\hat{f}\hat{w})| \le Cn^{-1}|\hat{E}(\hat{f}\hat{w})|.$$

We first turn to the proof of (4.6). Define, for $\hat{v}$, $\hat{w} \in P_k(\hat{I})$, a linear functional $\hat{\lambda}_{\hat{v}\hat{w}}$ on $W^{1,\infty}(\hat{I}\times\hat{I})$ by

$$\hat{\lambda}_{\hat{v}\hat{w}}(\hat{k}) = \hat{E}_{\hat{I}\times\hat{I}}(\hat{k}\hat{v}\hat{w}).$$

Then there exist positive constants C, independent of $\hat{v}$ and $\hat{w}$, such that

$$\begin{aligned}|\hat{\lambda}_{\hat{v}\hat{w}}(\hat{k})| &\le C\|\hat{k}\hat{v}\hat{w}\|_{0,\infty,\hat{I}\times\hat{I}} \le C\|\hat{k}\|_{0,\infty,\hat{I}\times\hat{I}}\|\hat{v}\|_{0,\infty,\hat{I}}\|\hat{w}\|_{0,\infty,\hat{I}}\\ &\le C\|\hat{k}\|_{1,\infty,\hat{I}\times\hat{I}}\|\hat{v}\|_{0,p,\hat{I}}\|\hat{w}\|_{0,p',\hat{I}},\end{aligned}$$

the last using norm-equivalence on the finite-dimensional space $P_k(\hat{I})$. Hence, $\hat{\lambda}_{\hat{v}\hat{w}}$ is a bounded linear functional on $W^{1,\infty}(\hat{I}\times\hat{I})$, with

$$\|\hat{\lambda}_{\hat{v}\hat{w}}\| \le C\|\hat{v}\|_{0,p,\hat{I}}\|\hat{w}\|_{0,p',\hat{I}}.$$

Moreover, since $\hat{v}$, $\hat{w} \in P_k(\hat{I})$, we have $\hat{k}\hat{v}\hat{w} \in P_{2k}(\hat{I}\times\hat{I})$ whenever $\hat{k} \in P_0(\hat{I}\times\hat{I})$, and so

$$\hat{\lambda}_{\hat{v}\hat{w}}(\hat{k}) = \hat{E}(\hat{k}\hat{v}\hat{w}) = 0 \qquad \forall\hat{k} \in P_0(\hat{I}\times\hat{I}).$$

Hence the Bramble-Hilbert lemma [**5**, Theorem 4.1.3] yields that there exists a positive constant C, such that for any $\hat{k} \in W^{1,\infty}(\hat{I}\times\hat{I})$ and any $\hat{v}$, $\hat{w} \in P_k(\hat{I})$, the estimate

$$(4.11)\qquad |\hat{E}_{\hat{I}\times\hat{I}}(\hat{k}\hat{v}\hat{w})| = |\hat{\lambda}_{\hat{v}\hat{w}}(\hat{k})| \le C|\hat{k}|_{1,\infty,\hat{I}\times\hat{I}}\|\hat{\lambda}_{\hat{v}\hat{w}}\| \le C|\hat{k}|_{1,\infty,\hat{I}\times\hat{I}}\|\hat{v}\|_{0,p,\hat{I}}\|\hat{w}\|_{0,p',\hat{I}}$$

holds. Since the grid sequence is quasi-uniform, we may use [**5**, Theorems 3.1.2 and 3.1.3] to see that

$$\begin{aligned}|\hat{k}|_{1,\infty,\hat{I}\times\hat{I}} &\le C\|B_{ab}\|\,|k|_{1,\infty,I_a\times I_b} \le C\,\mathrm{diam}(I_a\times I_b)|k|_{1,\infty,I_a\times I_b} \le Cn^{-2}|k|_{1,\infty,I_a\times I_b},\\ \|\hat{v}\|_{0,p,\hat{I}} &\le C\alpha_b^{-1/p}\|v\|_{0,p,I_b} \le Cn^{1/p}\|v\|_{0,p,I_b},\end{aligned}$$

and

$$(4.12)\qquad \|\hat{w}\|_{0,p',\hat{I}} \le C\alpha_a^{-1/p'}\|w\|_{0,p',I_a} \le Cn^{1/p'}\|w\|_{0,p',I_a}.$$

Hence (4.9), (4.11), the fact that $1/p+1/p'=1$, and the estimates above yield the desired result (4.6).

We next establish the estimate (4.7). Define, for $\hat{w}\in P_k(\hat{I})$, a linear functional $\hat{\lambda}_{\hat{w}}$ on $W^{\sigma,\infty}(\hat{I}\times\hat{I})$ by

$$\hat{\lambda}_{\hat{w}}(\hat{z})=\hat{E}_{\hat{I}\times\hat{I}}(\hat{z}\hat{w}).$$

Then there exist positive constants C, independent of $\hat{z}$ and $\hat{w}$, such that

$$|\hat{\lambda}_{\hat{w}}(\hat{z})|=|\hat{E}_{\hat{I}\times\hat{I}}(\hat{z}\hat{w})|\leq C\|\hat{z}\hat{w}\|_{0,\infty,\hat{I}\times\hat{I}}\leq C\|\hat{z}\|_{0,\infty,\hat{I}\times\hat{I}}\|\hat{w}\|_{0,\infty,\hat{I}}\leq C\|\hat{z}\|_{\sigma,\infty,\hat{I}\times\hat{I}}\|\hat{w}\|_{0,p',\hat{I}},$$

the last step again using norm-equivalence on the finite-dimensional space $P_k(\hat{I})$. Hence $\hat{\lambda}_{\hat{w}}$ is a bounded linear functional on $W^{\sigma,\infty}(\hat{I}\times\hat{I})$, with

$$\|\hat{\lambda}_{\hat{w}}\|\leq C\|\hat{w}\|_{0,p',\hat{I}}.$$

Moreover, since $\sigma-1\leq k+1$, we may use $\hat{w}\in P_k(\hat{I})$ to see that $\hat{z}\hat{w}\in P_{2k+1}(\hat{I})$ whenever $\hat{z}\in P_{\sigma-1}(\hat{I}\times\hat{I})$, and so

$$\hat{\lambda}_{\hat{w}}(\hat{z})=\hat{E}_{\hat{I}\times\hat{I}}(\hat{z}\hat{w})=0\qquad\forall\hat{z}\in P_{\sigma-1}(\hat{I}\times\hat{I}).$$

Hence the Bramble-Hilbert lemma yields that there exists a positive constant C, such that for any $\hat{w}\in P_k(\hat{I})$ and $\hat{z}\in W^{\sigma,\infty}(\hat{I}\times\hat{I})$, the estimate

$$|\hat{\lambda}_{\hat{w}}(\hat{z})|\leq C\|\hat{\lambda}_{\hat{w}}\||\hat{z}|_{\sigma,\infty,\hat{I}\times\hat{I}}\leq C|\hat{z}|_{\sigma,\infty,\hat{I}\times\hat{I}}\|\hat{w}\|_{0,p',\hat{I}}$$

holds. Now let $\hat{z}=\hat{k}\hat{v}$. Then norm-equivalence on the finite-dimensional space $P_{k-j}(\hat{I})$ yields

$$|\hat{v}|_{j,\infty,\hat{I}}=\|\hat{v}^{(j)}\|_{0,\infty,\hat{I}}\leq C\|\hat{v}^{(j)}\|_{0,p,\hat{I}}=C|\hat{v}|_{j,p,\hat{I}},$$

and so [5, (4.1.42)] yields

$$|\hat{k}\hat{v}|_{\sigma,\infty,\hat{I}\times\hat{I}}\leq C\sum_{j=0}^{\sigma}|\hat{k}|_{\sigma-j,\infty,\hat{I}\times\hat{I}}|\hat{v}|_{j,\infty,\hat{I}}\leq C\sum_{j=0}^{\sigma}|\hat{k}|_{\sigma-j,\infty,\hat{I}\times\hat{I}}|\hat{v}|_{j,p,\hat{I}}.$$

Thus

$$\begin{aligned}|\hat{E}_{\hat{I}\times\hat{I}}(\hat{k}\hat{v}\hat{w})|&=|\hat{\lambda}_{\hat{w}}(\hat{k}\hat{v})|\leq C|\hat{k}\hat{v}|_{\sigma,\infty,\hat{I}\times\hat{I}}\|\hat{w}\|_{0,p',\hat{I}}\\&\leq C\sum_{j=0}^{\sigma}|\hat{k}|_{\sigma-j,\infty,\hat{I}\times\hat{I}}|\hat{v}|_{j,p,\hat{I}}\|\hat{w}\|_{0,p',\hat{I}}.\end{aligned}\tag{4.13}$$

Since quasi-uniformity and [5, Theorems 3.1.2 and 3.1.3] yield

$$|\hat{k}|_{\sigma-j,\infty,\hat{I}\times\hat{I}}\leq C\left(\operatorname{diam}(I_a\times I_b)\right)^{\sigma-j}|k|_{\sigma-j,\infty,I_a\times I_b}\leq Cn^{-2(\sigma-j)}|k|_{\sigma-j,\infty,I_a\times I_b}$$

and

$$\|\hat{v}\|_{j,p,\hat{I}} \le C\alpha_b^{-1/p}\|v\|_{j,p,I_b} \le Cn^{1/p}\|v\|_{j,p,I_b},$$

the desired estimate (4.7) follows from (4.9), (4.12), (4.13), the estimates above, and the facts that $1/p + 1/p' = 1$ and $\sigma \le r$.

We finally turn to the proof of (4.8). For $\hat{w} \in P_k(\hat{I})$, define a linear functional $\hat{\lambda}_{\hat{w}}$ on $W^{\sigma,p}(\hat{I})$ by

$$\hat{\lambda}_{\hat{w}}(\hat{f}) = \hat{E}(\hat{f}\hat{w}).$$

Using the Sobolev embedding theorem and norm-equivalence on the finite-dimensional space $P_k(\hat{I})$, we see that there exist positive constants C, independent of $\hat{f}$ and $\hat{w}$, such that

$$|\hat{\lambda}_{\hat{w}}(\hat{f})| = |\hat{E}(\hat{f}\hat{w})| \le C\|\hat{f}\hat{w}\|_{0,\infty,\hat{I}} \le C\|\hat{f}\|_{0,\infty,\hat{I}}\|\hat{w}\|_{0,\infty,\hat{I}} \le C\|\hat{f}\|_{\sigma,p,\hat{I}}\|\hat{w}\|_{0,p',\hat{I}}.$$

Thus $\hat{\lambda}_{\hat{w}}$ is a bounded linear functional on $W^{\sigma,p}(\hat{I})$, with

$$\|\hat{\lambda}_{\hat{w}}\| \le C\|\hat{w}\|_{0,p',\hat{I}}.$$

Moreover, since $\hat{w} \in P_k(\hat{I})$ and $\sigma \le k+2$, we have $\hat{f}\hat{w} \in P_{2k+1}(\hat{I})$ whenever $\hat{f} \in P_{\sigma-1}(\hat{I})$, and so

$$\hat{\lambda}_{\hat{w}}(\hat{f}) = \hat{E}(\hat{f}\hat{w}) = 0 \qquad \forall \hat{f} \in P_{\sigma-1}(\hat{I}).$$

Thus the Bramble-Hilbert lemma yields that there exists a positive constant C, such that for any $\hat{f} \in W^{\sigma,p}(\hat{I})$ and any $\hat{w} \in P_k(\hat{I})$, the estimate

$$|\hat{E}(\hat{f}\hat{w})| = |\hat{\lambda}_{\hat{w}}(\hat{f})| \le C\|\hat{\lambda}_{\hat{w}}\|\,|\hat{f}|_{\sigma,p,\hat{I}} \le C|\hat{f}|_{\sigma,p,\hat{I}}\|\hat{w}\|_{0,p',\hat{I}} \tag{4.14}$$

holds. Using quasi-uniformity and [**5**, Theorems 3.1.2 and 3.1.3], we find

$$|\hat{f}|_{\sigma,p,\hat{I}} \le C\alpha_a^{-1/p}|f|_{\sigma,p,I_a}.$$

The desired bound (4.8) now follows easily by using this estimate, along with (4.10), (4.12), (4.14), and the facts that $1/p + 1/p' = 1$ and $\sigma \le r$.

We next give an estimate of the "global consistency error" by summing the estimates of the local consistency error. To do this, it is useful to define, for each positive integer n, an interpolation operator $\Pi_n : W^{r,p}(I) \to S_n$ by

$$\Pi_n v = \sum_{j=1}^{n} v(x_j)\, s_j,$$

where $\{ s_1, \ldots, s_n \}$ is a basis for S_n which is dual to the linear functionals which evaluate at $\{ x_1, \ldots, x_n \}$; that is, $s_1, \ldots, s_n \in S_n$ are chosen so that $s_i(x_j) = \delta_{ij}$ for $1 \le i, j \le n$.

LEMMA 4.3. *Let $\sigma = \min(k+2, r)$. There is a positive constant C, such that for any positive integer n, any v, $w \in S_n$, and any $f \in W^{r,p}(I)$, we have*

$$|B(v,w) - B_n(v,w)| \leq Cn^{-2}|k|_{1,\infty,I\times I}\|v\|_{0,p}\|w\|_{0,p'}, \tag{4.15}$$

$$|B(\Pi_n Sf, w) - B_n(\Pi_n Sf, w)| \leq Cn^{-\sigma}\|k\|_{r,\infty,I\times I}\|f\|_{r,p}\|w\|_{0,p'}, \tag{4.16}$$

and

$$|(f,w) - \tilde{f}_n(w)| \leq Cn^{-\sigma}\|f\|_{r,p}\|w\|_{0,p'}. \tag{4.17}$$

PROOF: We first show that (4.15) and (4.16) hold. By Lemma 4.2, there is a positive constant C, such that for any positive integer n and any v, $w \in S_n$,

$$\begin{aligned} |B(v,w) - B_n(v,w)| &\leq \sum_{a=1}^{m}\sum_{b=1}^{m}|E_{ab}(v,w)| \\ &\leq Cn^{-(\alpha+1)}\sum_{a=1}^{m}\sum_{b=1}^{m} a_{ab}\|v\|_{\beta,p,I_b}\|w\|_{0,p',I_a}, \end{aligned} \tag{4.18}$$

where either

$$\alpha = 2, \qquad a_{ab} = |k|_{1,\infty,I_a\times I_b}, \qquad \beta = 0 \tag{4.19}$$

or

$$\alpha = 0, \qquad a_{ab} = \|k\|_{r,\infty,I_a\times I_b}, \qquad \beta = r. \tag{4.20}$$

Since $m = \Theta(n)$ (by quasi-uniformity) and $a_{ab} \geq 0$, the discrete version of Hölder's inequality yields

$$\sum_{a=1}^{m}\sum_{b=1}^{m} a_{ab}\|v\|_{\beta,p,I_b}\|w\|_{0,p',I_a} \leq C\left[\max_{1\leq a,b\leq m} a_{ab}\right]\left[\sum_{b=1}^{m}\|v\|^p_{\beta,p,I_b}\right]^{1/p}\|w\|_{0,p'},$$

(with the obvious modification for the case $p = \infty$) which, when combined with (4.18), yields

$$|B(v,w) - B_n(v,w)| \leq Cn^{-\alpha}\left[\max_{1\leq a,b\leq m} a_{ab}\right]\left[\sum_{b=1}^{m}\|v\|^p_{\beta,p,I_b}\right]^{1/p}\|w\|_{0,p'}. \tag{4.21}$$

The result (4.15) now follows immediately from (4.19) and (4.21). In order to prove (4.16), let $v = \Pi_n Sf$ in (4.21). Since $P_k(I_b)$ is invariant under Π_n for each $b \in \{1, \ldots, m\}$, [**5**, Theorem 3.1.4] yields a bound of the form

$$\|Sf - \Pi_n Sf\|_{r,p,I_b} \leq C\|Sf\|_{r,p,I_b};$$

hence, the triangle inequality implies that there is a positive constant C such that

$$\|\Pi_n Sf\|_{r,p,I_b} \le C\|Sf\|_{r,p,I_b}\,.$$

This result, along with Theorem 2.1, implies that

$$\left[\sum_{b=1}^{m} \|\Pi_n Sf\|_{r,p,I_b}^p\right]^{1/p} \le C\|Sf\|_{r,p} \le C\|f\|_{r,p}\,.$$

Using this inequality with (4.20) and (4.21) gives the estimate (4.16).

Finally, we may use (4.8) and the discrete version of Hölder's inequality to see that

$$\begin{aligned}
|(f,w) - f_n(w)| &\le \sum_{a=1}^{m} |E_a(fw)| \le Cn^{-\sigma} \sum_{a=1}^{m} \|f\|_{r,p,I_a}\|w\|_{0,p',I_a} \\
&\le Cn^{-\sigma}\left[\sum_{a=1}^{m} \|f\|_{r,p,I_a}^p\right]^{1/p}\left[\sum_{a=1}^{m} \|w\|_{0,p',I_a}^{p'}\right]^{1/p'} \\
&= Cn^{-r}\|f\|_{r,p}\|w\|_{0,p'},
\end{aligned}$$

establishing (4.17).

Our next task is to establish the uniform weak coercivity of the bilinear forms $\{ B_n \}$. This is done by first establishing weak coercivity of the bilinear form B, and then using Lemma 4.3.

LEMMA 4.4. *The family $\{ B_n \}$ of bilinear forms is uniformly weakly coercive.*

PROOF: We first show that B is weakly coercive. That is, there is a positive integer n_0 and a positive constant β_0, such that for any $n \ge n_0$ and any $v \in S_n$, there is a nonzero $w \in S_n$ such that

$$|B(v,w)| \ge \beta_0\|v\|_{0,p}\|w\|_{0,p'}\,. \tag{4.22}$$

If $v = 0$, then (4.22) holds for *any* nonzero $w \in S_n$; hence it is no loss of generality to assume that v is nonzero. Since $1 < p \le \infty$, [9, (4.14.3) and (4.14.8)] implies that there is a nonzero $g \in L_{p'}(I)$ such that

$$|(v,g)| \ge \frac{1}{2}\|v\|_{0,p}\|g\|_{0,p'}. \tag{4.23}$$

Now choose $w \in S_n$ to be the finite element approximation of $(L*)^{-1}g$; that is,

$$B(s,w) = (s,g) \qquad \forall s \in S_n\,. \tag{4.24}$$

By (the adjoint version of) Theorem 4.1, w exists for all $n \ge n_0$, and

$$\|w\|_{0,p'} \le C\|g\|_{0,p'}. \tag{4.25}$$

Letting $\beta_0 = 1/(2C)$, we may use (4.23), (4.24), and (4.25) to see that

$$|B(v,w)| = |(v,g)| \geq \frac{1}{2}\|v\|_{0,p}\|g\|_{0,p'} \geq \beta_0\|v\|_{0,p}\|w\|_{0,p'}.$$

Finally, note that this inequality, along with the fact that v and g are nonzero, yields that $B(v,w) \neq 0$; since B is bilinear, this implies that $w \neq 0$.

Now let C be as in (4.15). Choose

$$n_1 = \max\left\{ n_0, \left\lceil \sqrt{\frac{C}{\beta_0}} \right\rceil + 1 \right\}$$

so that

$$\beta = \beta_0 - Cn_1^{-2}$$

is positive. Given $n \geq n_1$ and $v \in S_n$, choose nonzero $w \in S_n$ such that (4.22) holds. Then (4.15), (4.22), and the definitions of n_1 and β yield that

$$\begin{aligned}|B_n(v,w)| &\geq |B(v,w)| - |B(v,w) - B_n(v,w)| \geq (\beta_0 - Cn^{-2})\|v\|_{0,p}\|w\|_{0,p'} \\ &\geq \beta\|v\|_{0,p}\|w\|_{0,p'},\end{aligned}$$

as required.

We are now able to establish that the FEMQ φ_n^q is well-defined and to give sharp bounds on its error. We also show that the FEMQ is quasi-minimal iff (3.1) holds.

THEOREM 4.1. *There exists a positive integer n_1 such that the FEMQ φ_n^q is defined for all $n \geq n_1$. Moreover,*

$$e(\varphi_n^q) = \Theta(n^{-\mu}) \qquad \text{as} \qquad n \to \infty,$$

where (as in Theorem 4.1)

$$\mu = \min(k+1, r).$$

Hence, the FEMQ is a quasi-minimal error algorithm iff $k \geq r-1$.

PROOF: Let n_1 be as in Lemma 4.4; then Lemmas 4.1 and 4.4 imply that φ_n^q is defined for all $n \geq n_1$.

We next establish the upper bound $e(\varphi_n^q) = O(n^{-\mu})$. Let $f \in F$, so that $\|f\|_{r,p} \leq 1$, and let $n \geq n_1$. Since $P_k(I_b)$ is Π_n-invariant for $b \in \{1, \ldots, m\}$, [5, Theorem 3.1.4], the discrete version of the Hölder inequality, and Theorem 2.1 yield that

$$\|Sf - \Pi_n Sf\|_{0,p} \leq Cn^{-\mu}\|f\|_{r,p}.$$

Replacing s by $\Pi_n Sf$ in the bound of Lemma 4.1 (which can, at worst, increase the right-hand side of that bound), and using Lemmas 4.1–4.4 (along with the fact that $f \in F$), we have

$$\|Sf - \varphi_n^q(N_n^q f)\|_{0,p} \leq Cn^{-\mu}.$$

Since $f \in F$ is arbitrary, we may take the supremum over all such f to find the desired upper bound $e(\varphi_n^q) = O(n^{-\mu})$.

To establish the lower bound $e(\varphi_n^q) = \Omega(n^{-\mu})$, note that $\varphi_n^q(N_n^q f) \in S_n$, so that (as in the proof of the lower bound in Theorem 4.1), we have

$$e(\varphi_n^q) = \sup_{f \in F} \|Sf - \varphi_n^q(N_n^q f)\|_{0,p} \geq \sup_{f \in F} \inf_{s \in S_n} \|Sf - s\|_{0,p} \geq Cn^{-\mu},$$

completing the proof of the estimate $e(\varphi_n^q) = \Theta(n^{-\mu})$. The last part in the statement of the theorem follows immmediately from this estimate and Theorem 3.1

5. Complexity analysis

In this section, we discuss the complexity of finding ϵ-approximations to the solution of the Fredholm equation of the second kind, as well as the penalty for using the FEM when $k < r - 1$.

Let $\epsilon > 0$. An algorithm φ produces an ϵ-*approximation* to the problem if

$$e(\varphi) \leq \epsilon.$$

The *complexity* comp(ϵ) of an *algorithm* φ is defined via the model of computation discussed in [**20**, Chapter 5]. Informally, we assume that any linear functional can be evaluated with finite cost c_1, and that the cost of an arithmetic operation is unity.

Recall that φ_n denotes the finite element method of degree k using the finite element information N_n based on the finite element subspace S_n. Also, recall that φ_n^q denotes the finite element method with quadrature using the standard information N_n^q described in Section 4. Since the FEI N_n and the standard information N_n^q each contain n linearly independent linear functionals, we find that

$$\text{comp}(\varphi_n) \geq n\,c_1 \qquad \text{and} \qquad \text{comp}(\varphi_n^q) \geq n\,c_1.$$

Throughout the remainder of this paper, we assume that

$$\text{comp}(\varphi_n) = \Theta(n) \quad \text{and} \quad \text{comp}(\varphi_n^q) = \Theta(n) \qquad \text{as } n \to \infty. \tag{5.1}$$

REMARK 5.1. The assumption (5.1) is reasonable in either of two cases. In the first case, we actually assume the existence of an algorithm which can solve the linear system generated by the FEM, whose number of operations is linear in n, the size of the linear system. This condition holds in a number of special cases—finding such linear-time algorithms is still an open problem for the general case. (It is perhaps possible that the approaches of [**10**] and [**18**] may be used to transform the FEM or FEMQ into methods having roughly the same error as the original methods, yet whose linear systems can be solved in time which is linear in n.)

Alternatively, one may wish to make an assumption of *preconditioning*. That is, we assume that any computation which is independent of the right-hand side f is done in advance, and not counted when determining the number of operations required when approximating the solution u to the problem $Lu = f$. We make this notion of preconditioning more precise for the FEM (the case for the FEMQ being analogous). Recall that the FEM is a linear algorithm, i.e., it produces a linear approximation to the exact solution $u(x)$ having the form $\sum_{i=1}^{n}(f, s_i)g_i(x)$, where $g_1, \dots, g_n \in S_n$. Since $g_1, \dots g_n$ are independent of f, they may be determined in advance. (This precomputation may be especially efficient in the case where the problem $Lu = f$ is to be solved for many different right-hand sides f.) Hence, computing the value of the FEM at any point in I requires at most n multiplications and $n-1$ additions, once the n inner products $(f, s_1), \dots, (f, s_n)$ have been evaluated. Thus (5.1) holds for the FEM, if one uses preconditioning.

Let

$$\mathrm{FEM}(\epsilon) = \inf\{\,\mathrm{comp}(\varphi_n) : n \text{ is an index such that } e(\varphi_n) \le \epsilon\,\}$$

and

$$\mathrm{FEMQ}(\epsilon) = \inf\{\,\mathrm{comp}(\varphi_n^q) : n \text{ is an index such that } e(\varphi_n^q) \le \epsilon\,\}$$

denote the cost of finding an ϵ-approximation using the FEM and the FEMQ (respectively). From the results of Sections 3 and 4, along with (5.1), we find

THEOREM 5.1. *Let* $\mu = \min\{\,k+1, r\,\}$. *Then*

$$\mathrm{FEM}(\epsilon) = \Theta(\epsilon^{-1/\mu})$$

and

$$\mathrm{FEMQ}(\epsilon) = \Theta(\epsilon^{-1/\mu})$$

as $\epsilon \to 0$.

We now consider the Hilbert case $p = 2$. Let φ_n^s denote the spline algorithm using the FEI N_n. If we agree once more to accept the idea of preconditioning as discussed in Remark 5.1, we find

$$\mathrm{comp}(\varphi_n^s) = \Theta(n) \quad \text{as } n \to \infty. \tag{5.2}$$

We now let

$$\mathrm{SPLINE}(\epsilon) = \inf\{\,\mathrm{comp}(\varphi_n^s) : n \text{ is an index such that } e(\varphi_n^s) \le \epsilon\,\}$$

denote the cost of solving the problem using the spline algorithm (see Section 3). Using (5.2) and Theorem 3.3, we find

THEOREM 5.2. *In the Hilbert case* $p = 2$, *we have*

$$\mathrm{SPLINE}(\epsilon) = \Theta(\epsilon^{-1/r}) \text{ as } \epsilon \to 0.$$

We now wish to determine the minimal cost of solving the problem. Let

$$\mathrm{COMP}(\epsilon) = \inf\{\,\mathrm{comp}(\varphi) : \varphi \text{ is an algorithm for which } e(\varphi) \le \epsilon\,\}$$

denote the *problem complexity*, i.e., the inherent cost of solving the problem with error not exceeding ϵ. Using Theorem 3.1, (5.1), and Theorem 5.1, we find

THEOREM 5.3. *The problem complexity is*

$$\mathrm{COMP}(\epsilon) = \Theta(\epsilon^{-1/r}) \text{ as } \epsilon \to 0.$$

Hence, we may draw the following conclusions:

COROLLARY 5.1.

(i) *The spline algorithm using the FEI is always quasi-optimal in the Hilbert case.*

(ii) *The FEM and FEMQ are quasi-optimal iff $k \geq r - 1$. If $k < r - 1$, then for*

$$\lambda = \frac{1}{k+1} - \frac{1}{r} > 0,$$

the asymptotic penalty for using the FEM is

$$\frac{\mathrm{FEM}(\epsilon)}{\mathrm{COMP}(\epsilon)} = \Theta\left(\frac{1}{\epsilon^{\lambda}}\right)$$

and the asymptotic penalty for using the FEMQ is

$$\frac{\mathrm{FEMQ}(\epsilon)}{\mathrm{COMP}(\epsilon)} = \Theta\left(\frac{1}{\epsilon^{\lambda}}\right).$$

Hence when $k < r - 1$, the asymptotic penalty for using the FEM or FEMQ (rather than an optimal method) is unbounded, i.e.,

$$\lim_{\epsilon \to 0} \frac{\mathrm{FEM}(\epsilon)}{\mathrm{COMP}(\epsilon)} = \lim_{\epsilon \to 0} \frac{\mathrm{FEMQ}(\epsilon)}{\mathrm{COMP}(\epsilon)} = \infty.$$

Hence when k is too small for a given value of r, there is an infinite aysmptotic penalty for using the FEM or FEMQ. Based on examples similar to [**23**, Example 7.1], it is reasonable to suspect that this is not really an asymptotic result; we suspect that it is generally more costly to use an FEM or FEMQ whose degree is too small (rather than one of the proper degree), even for error criteria ϵ which are only "moderately" small, and hence of "practical" interest.

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