A Theoretical Analysis of the Conditions for Unambiguous Node Localization in Sensor Networks

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***Abstract*— In this paper we provide a theoretical foundation for the problem of network localization in which some nodes know their locations and other nodes determine their locations by measuring distances or bearings to their neighbors. Distance information is the separation between two nodes connected by a sensing/communication link. Bearing is the angle between a sensing/communication link and the x-axis of a node's local coordinate system. We construct grounded graphs to model network localization and apply graph rigidity theory and parallel drawings to test the conditions for unique localizability and to construct uniquely localizable networks. We further investigate partially localizable networks.**

I. Introduction

Location service is a basic service of many emerging computing/networking paradigms. In sensor networks, the sensor nodes need to know their locations in order to detect and record events, or to route packets using geometric-aware routing. In the case of generic ad hoc networks, position of the nodes is not always a requirement, but when it is available, more efficient implementation of network services is possible. For example, in pervasive computing knowing the locations of the computers and the printers in a building will allow a computer to send a printing job to the nearest printer [17].

In most cases, sensors are deployed without their position information known in advance, and there is no supporting infrastructure available to locate them after deployment. Sensor network protocols and algorithms must possess self-organizing capabilities [1]. It is necessary to find an alternative approach to identify the position of each sensor in wireless sensor networks after deployment. One method to determine the location of a node is manual configuration. However, this is unlikely to be feasible for any large-scale deployment or when nodes move often. Self positioning can be achieved by means of Global Positioning System (GPS). GPS has been widely used for positioning service. Although it is possible to find the position of each sensor in a wireless sensor network with the aid of GPS installed in all sensors, it is not practical to use because it is costly both in terms of hardware and power requirements. Furthermore, since GPS requires line-of-sight between the receiver and the satellites, it may not work well in buildings or in the presence of obstructions such as dense vegetation, foliage, or mountains blocking the direct view to the GPS satellites.

In the general model of wireless ad-hoc sensor network, there are usually some landmarks or nodes named *beacons* (also called anchor nodes), whose position information is known, within the area to facilitate locating all sensors in a sensor network. Those beacons have either GPS or they are manually configured. For the rest of the nodes two types of node capabilities are considered in this paper: distance measurements (also called ranging) and bearing measurements (also called angle of arrival (AOA)). Distance measurements provide the possibility for a node to measure distance to neighbors. If two nodes i and j have a sensing/communication link between each other as shown in Figure 1, then *bearing* information for i and j, denoted by θ_{ij} and θ_{ji} respectively, are the angles between the x-axis of each node's local coordinate system and the communication link (i, j). If each node uses its own coordinate system and is not aware of other nodes' coordinate systems, then nodes will not be able to reach a consensus to make use of the bearing information. In real implementations of bearing information, the information about a global coordinate system (x_G, y_G) is either known by all nodes or is transmitted from beacons to ordinary nodes [13]. This is done by passing "heading" information

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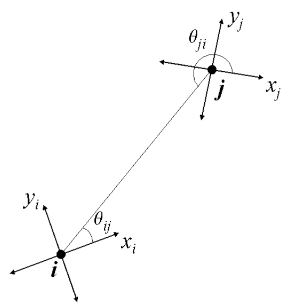

Fig. 1.

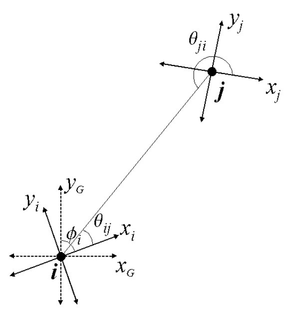

Fig. 2.

from one node to another. By *heading* is meant the angle between the y-axis of the global coordinate system and the x-axis of the node's local coordinate system. For example, ϕ_i is the heading of i in Figure 2. Once node i passes the information ϕ_i and θ_{ij} to node j, then node j can compute its heading by $\phi_j = \pi - (\theta_{ij} - \phi_i) + \theta_{ji}$. Once nodes know the global coordinate system, they can transform the bearing information measured in their local coordinate systems (θ_{ij} and θ_{ji}) into bearing information in the global coordinate system (Θ_{ij} and Θ_{ji}) as shown in Figure 3. We note that $\Theta_{ji} = \pi + \Theta_{ij}$.

For bearing measurement capability, each node in

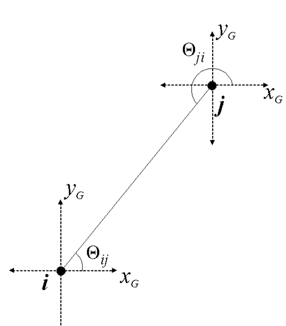

Fig. 3.

the network is assumed to have one main axis against which all bearings are reported and the capacity to estimate with a given precision the direction from which a neighbor is sending data. It is assumed that after the deployment, the axis of the node has an arbitrary, unknown heading. A node can infer its heading, if heading of one of its of the neighbors is known. If no compass is available in any node, but each node knows its position, heading can still be found [13].

Bearing capability is achieved by various technologies, some of which might be prohibitive in size and power consumption. A small form factor node that satisfies the conditions outlined above has been developed at MIT by the Cricket Compass project [14]. These nodes indicate that it is feasible to obtain bearing capability in a small package that would be appropriate for future pervasive ad hoc networks.

The process of computing the locations of the nodes is called network localization. It has been studied in other research areas such as robotics. In the context of sensor networks, some of the challenges are the size and the power of individual sensor nodes. Recently, novel schemes have been proposed to determine the locations of the nodes in a network where only some special nodes (beacons) know their locations. In these schemes, network nodes measure the distances or bearings to their neighbors and then try to determine their locations [13], [2], [12].

Although the designs of the previous schemes have demonstrated great engineering ingenuity, and their effectiveness is verified through extensive simulations, some fundamental questions have not been addressed; as a result, the previous schemes are mainly heuristic-based and a full theoretical foundation of network localization is still lacking. Specifically, we identified the following fundamental question in [5]: What are the conditions for unique network localizability? Although the network localization problem has already been studied extensively, the precise conditions under which the network localization problem is solvable are not known. In [5], we investigated sensor networks with distance information. Here we extend our analysis for sensor networks with both distance and bearing information. Furthermore we also investigate localization in subnetworks.

We address the unique localizability question using graph rigidity theory. More specifically, we propose grounded graphs. In these graphs, each vertex represents a network node, and two vertices in the graph are connected if the distance between the two is known, i.e., when the distance between the two nodes is measured or when the two nodes are beacon nodes and thus their distance is implicitly known. Given our construction of grounded graphs, we show that a network has a unique localization if and only if its corresponding grounded graph is generically globally rigid. By observing this connection, we are able to apply the results from the graph-rigidity literature to network localization. In [5] we proposed inductive sequences for constructing uniquely localizable networks, both in the plane and in 3- space. By following these sequences, a designer of a network can be assured that the constructed network is uniquely localizable, thus avoiding expensive trial-and-error procedures.

To reduce the computational and communication complexity of localization, which is important in settings such as sensor networks, we studied a class of graphs called trilateration graphs [5]. We showed that trilateration graphs are uniquely localizable and the locations of the nodes can be computed efficiently. Here we extend our previous work in [5] for networks with bearing information. Furthermore we analyze localization in subnetworks. A longer version of this paper is available as a technical report [6].

The rest of this paper is organized as follows. The specific network localization problem to be addressed is formulated in Section II. The concepts of rigidity and global rigidity are discussed in Section III. In Section IV, we study rigidity for sensor networks with bearing information. In Section V, we study localization for subnetworks. Our conclusion and future work are in Section VI.

II. Formulation

A. The Network Localization Problem

The "network localization problem with distance information" can be formulated as follows. One begins with a network $\mathbf{N}$ in real d-dimensional space $\{d = 2\}$ consisting of a set of $m > 0$ nodes labelled 1 through m which represent "beacons" together with $n-m > 0$ additional nodes labelled $m + 1$ through n which represent sensors. Each node is located at a fixed position in $\mathbb{R}^d$ and has associated with it a specific set of "neighboring" nodes. Although a node's neighbors are typically defined to be all other nodes within some specified range, other definitions could also be used. The essential property we will require in this paper is that the definition of a neighbor be a symmetric relation on $\{1, 2, \ldots, n\}$ in the sense that node j is a neighbor of node i if and only if node i is also a neighbor of node j. Under these conditions $\mathbf{N}$'s neighbor relationships can be conveniently described by an undirected graph $\mathbb{G}_\mathbf{N} = \{\mathcal{V}, \mathcal{E}_\mathbf{N}\}$ with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E}_\mathbf{N}$ defined so that (i, j) is one of the graph's edges just in case nodes i and j are neighbors. We assume throughout, that $\mathbb{G}_\mathbf{N}$ is a connected graph. The *network localization problem with distance information* is to determine the locations x_i of all sensor nodes in $\mathbb{R}^d$ given the graph of the network $\mathbb{G}_\mathbf{N}$, the positions of the beacons $x_j,\ j \in \{1, 2, \ldots, m\}$ in $\mathbb{R}^d$, and the distance $\delta_\mathbf{N}(i, j)$ between each neighbor pair $(i, j) \in \mathcal{E}_\mathbf{N}$.

The "network localization problem with bearing information" can be formulated in a similar way. The only difference is that instead of having the distance $\delta_\mathbf{N}(i, j)$ between each neighbor pair $(i, j) \in \mathcal{E}_\mathbf{N}$, we now have bearings $\beta_\mathbf{N}(i, j)$ between each neighbor pair $(i, j) \in \mathcal{E}_\mathbf{N}$. Note that there are two bearing information for each edge, one is measured by one of the nodes and the other is measured by the other node on the other side of the edge.

The network localization problem just formulated is said to be *solvable* if there is exactly one set of vectors $\{x_{m+1}, \ldots x_n\}$ in $\mathbb{R}^d$ which is consistent with the given data $\mathbb{G}_\mathbf{N}$, $\{x_1, x_2, \ldots, x_m\}$, and $\delta_\mathbf{N} : \mathcal{E}_\mathbf{N} \rightarrow \mathbb{R}$ (for bearings $\beta_\mathbf{N} : \mathcal{E}_\mathbf{N} \rightarrow [0, 2\pi)$). In this paper we will be concerned with "generic" solvability of the problem which means, roughly speaking, that the problem should be solvable not only for the given data but also for slightly perturbed but consistent versions of the given data. It is possible to make precise what generic solvability means as follows. Fix $\mathbb{G}_\mathbf{N}$ and let $e_1, e_2, \ldots, e_q$ denote the edges in $\mathcal{E}_\mathbf{N}$. Note that for any set of n points $y_1, y_2, \ldots, y_n$ in $\mathbb{R}^d$ there is a unique distance vector z whose kth component is the distance between y_i and y_j where $(i, j) = e_k$. This means that there is a well-defined function $f : \mathbb{R}^{nd} \rightarrow \mathbb{R}^{(md+q)}$ mapping $\{y_1, y_2, \ldots, y_n\} \longmapsto \{y_1, y_2, \ldots, y_m, z\}$. Solvability of the network localization problem is equivalent to f being injective $\{$at $\{x_1, x_2, \ldots, x_n\}\}$ in the sense that the only set of points $\{y_1, y_2, \ldots, y_n\} \in \mathbb{R}^{nd}$ for which $f(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_n)$ is $\{y_1, y_2, \ldots, y_n\} = \{x_1, x_2, \ldots, x_n\}$. In this context it is natural to say that the network localization problem is *generically solvable* at $\{x_1, x_2, \ldots, x_n\}$ if it is solvable at each point in an open neighborhood of $\{x_1, x_2, \ldots, x_n\}$. In other words, the localization problem is solvable at $\{x_1, x_2, \ldots, x_n\}$ if there is an open neighborhood of $\{x_1, x_2, \ldots, x_n\}$ on which f is an injective function.

B. Point Formations

To study the solvability of the network localization problem, we reformulate the problem in terms of a "point formation." As we shall see, the point formation relevant to the network localization problem has associated with it a graph with the same vertices as $\mathbb{G}_{\mathbf{N}}$ but with a slightly larger edge set which includes "links" or edges from every beacon to every other. It is a property of this graph rather than $\mathbb{G}_{\mathbf{N}}$ which proves to be central to solvability of the localization problem under consideration. We begin by reviewing the point formation concept.

By a d-dimensional *point formation* at $p \triangleq \text{column } \{p_1, p_2, \ldots, p_n\}$, written $\mathbb{F}_p$, is meant a set of n points $\{p_1, p_2, \ldots, p_n\}$ in $\mathbb{R}^d$ together with a set $\mathcal{L}$ of k *links*, labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$; the *length* of link (i, j) is the Euclidean distance between point p_i and p_j. The idea of a point formation is essentially the same as the concept of a "framework" studied in mathematics [15], [18] as well as within the theory of structures in mechanical and civil engineering. For our purposes, a point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ provides a natural high-level model for an n-node sensor network in real 2- or 3- dimensional space. In this context, the points p_i represent the positions of nodes {i.e., both sensors and beacons} in $\mathbb{R}^d$ and the links in $\mathcal{L}$ label those specific node pairs whose inter-node distances are given.

Thus for the sensor network discussed above, $\mathcal{L}$ would consist of not only all pairs in $\mathcal{E}_{\mathbf{N}}$, but also all additional beacon pairs $(i, j),\ i, j \in \{1, 2, \ldots, m\}$ since the distances between pairs of beacons are uniquely specified by their position vectors which are given.

A point formation $\mathbb{F}_p \triangleq (p, \mathcal{E})$ provides a way of representing a formation of n nodes. $p \triangleq \{p_1, p_2, \ldots, p_n\}$ and the points p_i represent the positions of nodes in $\mathbb{R}^d$ $\{d = 2 \text{ or } 3\}$ where i is an integer in $\{1, 2, \ldots, n\}$ and denotes the labels of nodes. $\mathcal{E}$ is the set of "maintenance links," labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$. The *maintenance links* in $\mathcal{E}$ correspond to constraints between specific nodes, such as distances and bearings, which are to be maintained over time by using sensing/communication links between certain pairs of nodes. Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G}_{\mathbb{F}_p} \triangleq (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$, which is the set of labels of nodes, and edge set $\mathcal{E}$. We will denote the set of maintenance links with distance constraints by $\mathcal{L}$, the set of maintenance links with bearing constraints by $\mathcal{B}$. A formation with distance constraints can be represented by $(\mathcal{V}, \mathcal{L}, f)$ where $f : \mathcal{L} \longmapsto \mathbb{R}$. Each maintenance link $(i, j) \in \mathcal{L}$ is used to maintain the distance $f((i, j))$ between certain pairs of nodes fixed. A formation with bearing constraints can be represented by $(\mathcal{V}, \mathcal{B}, g)$ where $g : \mathcal{B} \longmapsto [0, 2\pi)$. Each maintenance link $(i, j) \in \mathcal{B}$ is used to maintain the bearing $g((i, j))$ of the line joining certain pairs of nodes fixed with respect to a reference coordinate system. Let us note that the distance function of $\mathbb{F}_p$ is the same as the distance function of any point formation $\mathbb{F}_q$ with the same graph as $\mathbb{F}_p$ provided q is *congruent* to p in the sense that there is a distance preserving map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $T(q_i) = p_i, i \in \{1, 2, \ldots, n\}$. In the sequel we will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are *congruent* if they have the same graph and if q and p are congruent. It is clear that $\mathbb{F}_p$ is uniquely determined by its graph and distance function *at most* up to a congruence transformation. A formation which is *exactly* determined up to congruence by its graph and distance function is called "globally rigid." More precisely, a d-dimensional point formation $\mathbb{F}_p$ is said to be *globally rigid* if each d-dimensional point formation $\mathbb{F}_q$ with the same graph and distance function as $\mathbb{F}_p$ is congruent to $\mathbb{F}_p$. It is clear then any formation whose graph is complete is globally rigid. The following simple generalizations of this fact provide sufficient conditions for global rigidity which are especially relevant to the network localization problem.

Lemma 1.

Let $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ be a formation in $\mathbb{R}^2$ which contains three points p_a, p_b, and p_c which are not co-linear. Suppose that the formation consisting of these three points and all links from $\mathbb{F}_p$ which connect pairs of these three points, has a graph which is complete. Then $\mathbb{F}_p$ is globally rigid if and only if it is the only n-point formation in $\mathbb{R}^2$ which contains these three points and has link set $\mathcal{L}$.

These properties are direct consequences respectively of the fact that the identity on $\mathbb{R}^2$ is the only distance preserving map $T : \mathbb{R}^2 \rightarrow R^2$ which leaves p_a, p_b, and p_c unchanged. A proof of the lemma will not be given.

C. Solvability of the Network Localization Problem

With previous definition of point formations, we can now restate the network localization problem in terms of its associated point formation $\mathbb{F}_x$. In the present context, the problem is to determine $\mathbb{F}_x$, given the graph and distance function of $\mathbb{F}_x$ as well as the beacon position vectors $x_1, x_2, \ldots, x_m$. Solvability of the problem demands that $\mathbb{F}_x$ be globally rigid; for if $\mathbb{F}_x$ were not globally rigid it would be impossible to determine $\mathbb{F}_x$ up to congruence, let alone to determine it uniquely. Assuming $\mathbb{F}_x$ is globally rigid, solvability of the sensor network localization problem reduces to making sure that the group of transformations T which leaves the set $\{x_1, x_2, \ldots, x_m\}$ unchanged – namely distance preserving transformations $T : \mathbb{R}^d \to \mathbb{R}^d$ for which $T(x_i) = x_i, \; i \in \{1, 2, \ldots, m\}$ – also leaves unchanged the set $\{x_{m+1}, \ldots, x_n\}$. About the easiest way to guarantee this in $\mathbb{R}^2$ is to require $\{x_1, x_2, \ldots, x_m\}$ to contain three vectors $x_{i_1}, x_{i_2}, x_{i_3}$ which are not co-linear; for if this is so, then the only distance preserving transformation which leaves $\{x_1, x_2, \ldots, x_m\}$ unchanged is the identity map on $\mathbb{R}^2$. Similarly, if in $\mathbb{R}^3$ $\{x_1, x_2, \ldots, x_m\}$ contains at least four vectors which are not co-planar, then T will again be an identity map, in this case on $\mathbb{R}^3$. We summarize.

Theorem 2. *Let* $\mathbf{N}$ *be a sensor network in* $\mathbb{R}^d$, $\{d = 2 \text{ or } 3\}$, *consisting of* $m > 0$ *beacons located at positions* $x_1, x_2, \ldots, x_m$ *and* $n - m > 0$ *sensors located positions* $x_{m+1}, \ldots, x_n$. *Suppose that for the case* $d = 2$ *there are at least three beacons which are not positioned in a single line. Let* $\mathbb{F}_x$ *denote the point formation whose points are at* $x_1, x_2, \ldots, x_n$ *and whose links are those labelling all neighbor pairs and all beacon pairs in* $\mathbf{N}$. *Then for both* $d = 2$ *and* $d = 3$ *the sensor network localization problem is solvable if and only if* $\mathbb{F}_x$ *is globally rigid.*

III. Rigidity and Global Rigidity

In the previous section, we established that under certain mild conditions, the solvability of the network localization problem is equivalent to the "global rigidity" of a suitably defined point formation. We study rigidity and global rigidity in this section.

One way of visualizing rigidity is to imagine a collection of rigid bars connected to one another by idealized ball joints, which is called a bar-joint framework. By an idealized ball joint we mean a connection between a collection of bars which imposes only the restriction that the bars share common endpoints. Now, can the bars and joints be moved in a continuous manner without changing the lengths of any of the bars, where translations and rotations do not count? If so, the framework is flexible; if not, it is rigid. (Precise definitions will appear in the sequel.) In a bar-joint framework, the length of a bar imposes a distance constraint for both end-joints. This is the same situation in a formation where two nodes connected by a sensing/communication link are mutually affected by the information conveyed by this link. For example, if two nodes connected by a sensing/communication link are set to maintain a ten meter distance between each other, then both nodes perform action to maintain this distance. In the graph theoretic setting, the edge corresponding to this link is denoted by an undirected edge.

A *trajectory* of a formation is a continuously parameterized one-parameter family of curves $(q_1(t), q_2(t), \ldots, q_n(t))$ in $\mathbb{R}^{nd}$ which contain p and on which for each t, $\mathbb{F}_{q(t)}$ is a formation with the same measured values under f, g. A *rigid motion* is a trajectory along which point formations contained in this trajectory are congruent to each other. We will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_r$, where $p, r \in q(t)$, are congruent if they have the same graph and if p and r are congruent. p is *congruent* to r in the sense that there is a distance-preserving map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(r_i) = p_i, i \in \{1, 2, \ldots, n\}$. If rigid motions are the only possible trajectories then the formation is called *rigid*; otherwise it is called *flexible* [4].

A *parallel rigid motion* is a trajectory along which point formations contained in this trajectory are translations or dilations of each other. Two point formations $\mathbb{F}_p$ and $\mathbb{F}_r$ are *parallel* if they have the same graph and their corresponding maintenance links are parallel to each other. If parallel rigid motions are the only possible trajectories then the formation is called *parallel rigid*, otherwise *parallel flexible*.

As we've already stated, a d-dimensional point formation $\mathbb{F}_p$ is globally rigid if each d-dimensional point formation $\mathbb{F}_q$ with the same graph and distance function as $\mathbb{F}_p$ is congruent to $\mathbb{F}_p$. In order to clearly understand what global rigidity means we need several other concepts whose roots can be found in the classical theory of structures.

A. Rigidity

Let $\mathbb{F}_p$ be d-dimensional point formation. Even though the nodes in the networks we are considering are in fixed positions, it is useful to consider trajectories of such formations. By a *trajectory* of $\mathbb{F}_p$ is meant a continuously parameterized, one-parameter family of points $\{q(t) : t \geq 0\}$ in $\mathbb{R}^{nd}$ which contains p. It is natural to say that such a formation undergoes *rigid motion* along a trajectory $q([0, \infty)) \triangleq$ column$\{q_1(t), q_2(t), \dots, q_n(t)\} : t \geq 0\}$ if the Euclidean distance between each pair of points $q_i(t)$ and $q_j(t)$ remains constant all along the trajectory. Let us note that $\mathbb{F}_p$ undergoes rigid motion along a trajectory $q([0, \infty))$ just in case each pair of points $q(t_1), q(t_2) \in q([0, \infty))$ are congruent. The set of points $\mathcal{M}_p$ in $\mathbb{R}^{dn}$ which are congruent to p is known to be a smooth manifold [15]. It is clear that any trajectory along which $\mathbb{F}_p$ undergoes rigid motion must lie completely within $\mathcal{M}_p$; conversely any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$ is one along which $\mathbb{F}_p$ undergoes rigid motion.

A formation $\mathbb{F}_p$ is *rigid* if rigid motion is the only kind of motion it can undergo along any trajectory on which the lengths of all links in $\mathcal{L}$ remain constant. Thus if $\mathbb{F}_p$ is rigid, its points "remain in formation" provided that the lengths of all of the formation's links do not change as the formation moves. As we've already noted, for sensor localization we need networks whose point formations are uniquely determined up to congruence by their graphs and distance functions. Unfortunately rigidity is not a strong enough property of a formation to ensure that this is so. In other words it is possible to construct two rigid formations $\mathbb{F}_p$ and $\mathbb{F}_q$ which both have the same graph and distance function, but are not congruent. The subtly here stems from the fact that rigidity of $\mathbb{F}_p$ stipulates that only those formations encountered on trajectories containing $\mathbb{F}_p$ be congruent to $\mathbb{F}_p$. Unfortunately there are formations with the same graph and distance function as $\mathbb{F}_p$ which cannot be reached from $\mathbb{F}_p$ on any trajectory; such formations are typically not congruent to $\mathbb{F}_p$. From a different perspective, a rigid formation is a formation which is impossible to *continuously* deform while holding fixed the lengths of all of its links. There are examples of rigid formations which can indeed be deformed, but not continuously; such formations are rigid but not globally rigid. In the end, the key feature which distinguishes globally rigid formations from all others including those which are merely rigid, is that the former cannot be deformed by any means whatever, continuous or not, whereas the latter always can.

An example of a rigid formation which can be deformed discontinuously, is shown in Figure 4(a). Observe that a discontinuous deformation can be obtained by reflecting the triangle formed by points a, b, and c about the line determined by points a and b. The resulting rigid formation is shown in Figure 4(b).

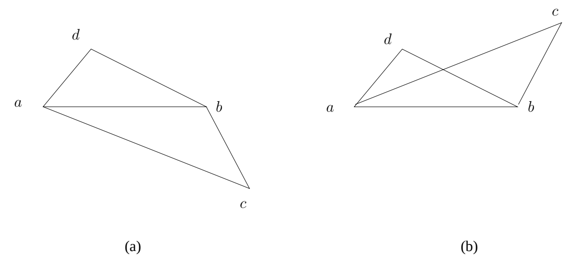

Fig. 4. Two rigid formations with the same graph and distance function

Adding a link from point c to d in Figure 4(a) would make the formation globally rigid. An example of a globally rigid formation whose graph is not complete is shown in Figure 5.

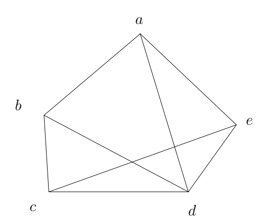

Fig. 5. A globally rigid formation

B. Conditions for Rigidity

The question of whether or not a given formation is rigid has been studied for a long time [15], [18], [11]. One starts by examining what happens to a given formation $\mathbb{F}_p = (\{p_1, p_2, \dots, p_n\}, \mathcal{L})$, along trajectory $\{q_1(t), q_2(t), \dots, q_n(t)\} : t \geq 0\}$ on which the Euclidean distances $\delta(i, j) \triangleq ||p_i - p_j||$ between pairs of points (q_i, q_j) for which (i, j) is a link, are constant. Thus along such a trajectory $(q_i - q_j)'(q_i - p_j) = \delta(i, j)^2, \ (i, j) \in \mathcal{L}, \ t \geq 0$. Assuming a smooth trajectory, these equations can be differentiated to get $(q_i - q_j)'(\dot{q}_i - \dot{q}_j) = 0, \ (i, j) \in \mathcal{L}, \ t \geq 0$. These equations can be rewritten in matrix form as

$$R(q)\dot{q} = 0 \tag{1}$$

where $\dot{q} =$ column $\{\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n\}$, and $R(q)$ is a specially structured $m \times dn$ array called a *rigidity matrix*.

Example 3. *Consider a planar point formation $\mathbb{F}_p$ shown in Figure 6. This has a rigidity matrix as shown in Table I.*

If adding a link (i, j) does not increase the rank of the rigidity matrix, then we call (i, j) an *implicit link* (*implicit edge* in the underlying graph).

Let $\mathcal{M}_p$ be the manifold of points congruent to p. Because any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$, is one along which $\mathbb{F}_p$ undergoes rigid motion, (1) automatically holds along any trajectory which lies within $\mathcal{M}_p$. From this, it follows that the tangent space to $\mathcal{M}_p$ at p, written $\mathcal{T}_p$, must be contained in the kernel of $R(p)$. If the points $p_1, p_2, \ldots, p_n$ are in general position (which means that the points $p_1, p_2, \ldots, p_n$ do not lie on any hyperplane in $\mathbb{R}^n$), then $\mathcal{M}_p$ is $n(n+1)/2$ dimensional since it arises from the $n(n-1)/2$-dimensional manifold of orthogonal transformations of $\mathbb{R}^n$ and the n-dimensional manifold of translations of $\mathbb{R}^n$ [15]. Thus $\mathcal{M}_p$ is 6-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^3$, and 3-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^2$. We have $\text{rank } R(p) = nd-$ dimension kernel $R(q) \leq nd - n(n+1)/2$. We have the following theorem [15], [19]:

Theorem 4. *Assume $\mathbb{F}_p$ is a formation with at least d points in d-dimensional space $\{d = 2$, or $3\}$ where $\text{rank } R(p) = \max\{\text{rank } R(x) : x \in \mathbb{R}^d\}$. $\mathbb{F}_p$ is rigid in $\mathbb{R}^d$ if and only if*

$$\text{rank } R(p) = \begin{cases} 2n-3 & \textit{if } d = 2, \\ 3n-6 & \textit{if } d = 3. \end{cases}$$

This theorem leads to the notion of the "generic" behavior of rigidity. When the rank is less than the maximum, the formation may still be rigid. However this type of rigidity lacks the generic behavior and thus is not addressed in this paper.

It is possible to characterize generic rigidity in terms of the "generic rank" of R where by R's *generic* or maximal rank we mean the largest value of $\text{rank}\{R(q)\}$ as q ranges over all values in $\mathbb{R}^{nd}$. The following theorem is due to Roth [15].

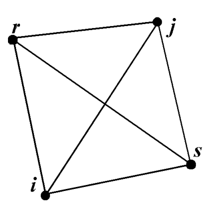

Fig. 6. A planar point formation.

Theorem 5. *A formation $\mathbb{F}_p$ is generically rigid if and only if*

$$\text{generic rank } \{R(p)\} = \begin{cases} 2n-3 & \textit{if } d = 2, \\ 3n-6 & \textit{if } d = 3. \end{cases}$$

To understand this type of rigidity, it is useful to observe that the set of points p that satisfy the condition $\text{rank } R(p) = \max\{\text{rank } R(x) : x \in \mathbb{R}^d\}$ is a dense open subset of $\mathbb{R}^{nd}$ [15]. Thus, a generically rigid point formation $\mathbb{F}_p$ is rigid for almost all points in the neighborhood of points about p in $\mathbb{R}^{dn}$. The concept of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}_p$ but examines how well the rigidity of formations can be judged by knowing the vertices and their incidences, in other words, by knowing the underlying graph. For this reason, it is a desirable specialization of the concept of a "rigid formation" for our purposes. We have the following theorem for a generically rigid graph [18]:

Theorem 6. *The following are equivalent:*

1) *a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in d-dimensional space ($d = 2, 3$);*
2) *for some p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ is generically rigid;*
3) *for almost all p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ is generically rigid.*

A point formation $\mathbb{F}_p$ is *strongly generically rigid* if it is generically rigid and if $\text{rank } R(p) = \text{generic rank } \{R\}$. Hence, a strongly generically rigid formation is rigid and it remains rigid under small perturbations. This is the type of rigidity that is useful for our purposes.

It is easy to see that all the entries in $R(p)$ are polynomial $\{$actually linear$\}$ functions of p. Because of this, the values of p for which the rank of $R(p)$ is below its maximum value, form a proper algebraic set in $\mathbb{R}^{dn}$. This and Theorem 5 imply that if $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ is generically rigid, then so is $\mathbb{F}_q = (\{q_1, q_2, \ldots, q_n\}, \mathcal{L})$ for all values of q not in the aforementioned proper algebraic set. Said differently, if $\mathbb{F}_p$ is generically rigid, then so is "almost every" other formation in $\mathbb{R}^d$ with the same set of links.

As noted above the concept of generic rigidity does not depend on the precise distances between the points in $\mathbb{F}_p$. It is perhaps not surprising then, that generic rigidity can be characterized in terms of the graph of $\mathbb{F}_p$ without any reference to $\mathbb{F}_p$'s actual points or distance function. To do this let us agree to say

$R(p)$	i		j		r		s	
(i,j)	$x_i - x_j$	$y_i - y_j$	$x_j - x_i$	$y_j - y_i$	0	0	0	0
(i,r)	$x_i - x_r$	$y_i - y_r$	0	0	$x_r - x_i$	$y_r - y_i$	0	0
(i,s)	$x_i - x_s$	$y_i - y_s$	0	0	0	0	$x_s - x_i$	$y_s - y_i$
(j,r)	0	0	$x_j - x_r$	$y_j - y_r$	$x_r - x_j$	$y_r - y_j$	0	0
(j,s)	0	0	$x_j - x_s$	$y_j - y_s$	0	0	$x_s - x_j$	$y_s - y_j$
(r,s)	0	0	0	0	$x_r - x_s$	$y_r - y_s$	$x_s - x_r$	$y_s - y_r$

TABLE I

RIGIDITY MATRIX EXAMPLE FOR DISTANCES

that a simple graph $\mathbb{G} \triangleq \{\mathcal{V}, \mathcal{L}\}$ with n vertices is *generically rigid* in $\mathbb{R}^d$ if there is an open dense set of points $p \in \mathbb{R}^{dn}$ at which $\mathbb{F}_p$ is a rigid formation with link set $\mathcal{L}$. The following theorem settles the generic rigidity question for $d = 2$ in strictly graph theoretic terms.

Theorem 7 (Laman [11]). *A graph $\mathbb{G} \triangleq \{\mathcal{V}, \mathcal{L}\}$ with n vertices is generically rigid in $\mathbb{R}^2$ if and only if $\mathcal{L}$ contains a subset $\mathcal{E}$ consisting of $2n - 3$ edges with the property that for any nonempty subset $\bar{\mathcal{E}} \subset \mathcal{E}$ the number of edges in $\mathcal{E}$ cannot exceed $2j - 3$ where j is the number of vertices of $\mathbb{G}$ which are endpoints of edges in $\bar{\mathcal{E}}$.*

The generalization of Laman's theorem to higher dimensions, including most especially $d = 3$ has prove quite elusive. At present this is the most general result known for characterizing generic rigidity in graph theoretic terms.

C. Conditions for Global Rigidity

Let us agree to say that a formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ of n points in $\mathbb{R}^d$ is *generically globally rigid* if for each q in some open neighborhood of p in $\mathbb{R}^{dn}$, formation $\mathbb{F}_q = (\{q_1, q_2, \ldots, q_n\}, \mathcal{L})$ is globally rigid. Since generic global rigidity implies global rigidity, it is clear that generic global rigidity of $\mathbb{F}_x$ is a sufficient condition for the conclusion of Theorem 2 to hold. There is a graph-theoretic characterization of generic global rigidity for 2-dimensional formations analogous to the characterization of generic rigidity provided by Laman's theorem {i.e., theorem 7}. To explain the result we need a few more concepts.

Recall that a connected graph $\mathbb{G}$ is *k-connected* if it is possible to obtain from it a new graph with at least two distinct connected components by removing at least one set of k vertices from $\mathbb{G}$ along with all of those edges of $\mathbb{G}$ which are incident on the k vertices being removed. The k-connectivity of a complete graph with n vertices is defined to be $n - 1$. A graph $\mathbb{G}$ which is generically rigid in $\mathbb{R}^d$ is *redundantly rigid* in $\mathbb{R}^d$ if removal of any single edge results in a graph which is also generically rigid in $\mathbb{R}^d$. Finally, a connected simple graph $\mathbb{G} = \{\mathcal{V}, \mathcal{L}\}$ with n vertices is *generically globally rigid* in $\mathbb{R}^d$ if there is an open dense set of points $p \in \mathbb{R}^{dn}$ at which $\mathbb{F}_p$ is a globally rigid formation with link set $\mathcal{L}$. The following recent result settles the generic global rigidity question for $d = 2$ in graph theoretic terms.

Theorem 8 ([10]). *A connected simple graph $\mathbb{G}$ with $n \geq 4$ vertices is generically globally rigid in $\mathbb{R}^2$ if and only if it is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

Let us note that to actually carry out a test to decide whether or not a given graph $\mathbb{G}$ is generically globally rigid in $\mathbb{R}^2$, one must establish that it is both 3-connected and redundantly rigid in $\mathbb{R}^2$. Various tests for 3-connectivity are known and we refer the reader to [9] for details including measures of the complexity of the tests involved. Tests for redundant rigidity in $\mathbb{R}^2$ have recently been derived [8] based on variants of Laman's theorem [11].

Much like the situation with generic rigidity, the generalization of Theorem 8 to higher dimensions does not yet exist. Nonetheless it is possible to derive various sufficient condition for a graph to be generically globally rigid in spaces of dimension greater than 2. The following result [5] is an example of this which gives a sufficient condition for generic global rigidity in both $\mathbb{R}^2$ and $\mathbb{R}^3$. The theorem extends to $\mathbb{R}^3$ earlier work by Jackson-Jordan [10] who establishes essentially the same result for $\mathbb{R}^2$.

Theorem 9. *([5]) Fix $d \in \{2, 3\}$ and let $\mathbb{F}_p$ be a formation in $\mathbb{R}^d$ whose graph is connected and consists of $n \geq d + 1$ vertices and link set $\mathcal{L}$.*

Suppose that there exists a set of $n-d$ formations $\mathbb{F}(1), \mathbb{F}(2), \ldots, \mathbb{F}(n-d)$ in $\mathbb{R}^d$ such that

1) *$F(1)$ contains exactly $d+1$ points, all from $\mathbb{F}_p$, which are not co-linear if $d=2$ and not co-planer if $d=3$.*
2) *$F(1)$ has a complete graph.*
3) *For $i \in \{2, \ldots, n-d\}$, $\mathbb{F}(i)$ is obtained from $\mathbb{F}(i-1)$ by adding to $\mathbb{F}(i-1)$ exactly one new point from $\mathbb{F}_p$ together with $d+1$ incident edges from $\mathcal{L}$; the $d+1$ points in $\mathbb{F}(i-1)$ upon which the added edges are incident, are not co-linear if $d=2$ and not co-planer if $d=3$.*
4) *$\mathbb{F}_p = \mathbb{F}(n-d)$.*

Then $\mathbb{F}_p$ is generically globally rigid in $\mathbb{R}^d$.

The utility of this sufficient condition is that it enables us to devise a provably correct sequential network localization algorithm using "tri-laterialization" which can be executed in a distributed manner. Theorem 9 is a simple consequence of the following lemma [5].

Lemma 10. *Let $\mathbb{F}_p$ be a globally rigid formation in $\mathbb{R}^2$ with three points p_a, p_b, p_c which are not co-linear. Let $\bar{\mathbb{F}}$ be the formation which results by adding to the point and link sets of $\mathbb{F}_p$ respectively, one new point $\bar{p} \in \mathbb{R}^2$ and links from this point to p_a, p_b, and p_c. Then $\bar{\mathbb{F}}$ is a globally rigid formation.*

IV. Rigidity for Networks with Bearing Information

The analysis in the previous section applies to sensor networks with distance information. Now we proceed to investigate global rigidity for networks with bearing information. Before proceeding further, we introduce "parallel drawings." Parallel drawings have been studied in rigidity and plane configurations in computer-aided design (CAD). A *plane configuration* is a collection of geometric objects such as points, line segments, and circular arcs in the plane, together with constraints on and between these objects [16]. Two point formations on the same graph are *parallel drawings* if corresponding edges are parallel. Parallel drawings, used by engineering draftsmen in the nineteenth century, have reappeared in a number of branches of discrete geometry [19].

Given a point formation $\mathbb{F}_r$, we are interested in parallel drawings $\mathbb{F}_s$ in which $s_i - s_j$ is parallel to $r_i - r_j$ for all $(i,j) \in \mathcal{E}$. Using the operator $(.)^\perp$, for turning a plane vector by $\frac{\pi}{2}$ counterclockwise, these constraints can be written:

$$(r_i - r_j)^\perp \cdot (s_i - s_j) = 0. \tag{2}$$

Each such constraint is a parallel drawing constraint. This gives a system of $|\mathcal{E}|$ homogeneous linear equations, and a parallel drawing is a solution of this system.

We have the following proposition [7].

Proposition 11. *A bearing constraint can be written as a parallel drawing constraint.*

For every link with a bearing constraint in the point formation, it is now straightforward to write

$$(p_i - p_j)^\perp \cdot (q_i(t) - q_j(t)) = 0, \quad (i,j) \in \mathcal{B}, \quad t \geq 0. \tag{3}$$

This gives a system of $|\mathcal{B}|$ homogenous linear equations. A solution of this system is called a *parallel point formation*.

Central to the development in the rest of this section will be the use of parallel drawings of configurations [16]. Given a point formation in 2-dimensional space with bearing constraints $\mathbb{F}_p$, we are interested in parallel point formations $\mathbb{F}_r$ in which $r_i - r_j$ is parallel to $p_i - p_j$ for all $(i,j) \in \mathcal{B}$. Trivially parallel point formations are translations and dilations of the original point formation, including the parallel point formation in which all points are coincident. All others are non-trivial. For example, Figure 7b shows a translation of the point formation in Figure 7a; and Figure 7c and Figure 7d are dilations of the point formation in Figure 7a. In particular Figure 7c is a contraction and Figure 7d is an expansion. Figure 7e shows a non-trivial parallel point formation of Figure 7a. A point formation with bearing constraints is called *parallel rigid* if all parallel point formations are trivially parallel. Otherwise it is called *flexible*. For example, the point formation in Figure 7a is flexible. On the other hand, Figure 7f shows a parallel rigid point formation.

Taking the derivative of (3) (recall that p is a fixed point set and $q(t)$ is time varying in (3)), we obtain

$$(p_i - p_j)^\perp \cdot (\dot{q}_i(t) - \dot{q}_j(t)) = 0, \quad (i,j) \in \mathcal{D}, \quad t \geq 0 \tag{4}$$

These equations can be rewritten in matrix form as

$$R(p)\dot{q} = 0 \tag{5}$$

where $\dot{q} = \text{column}\ \{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\}$.

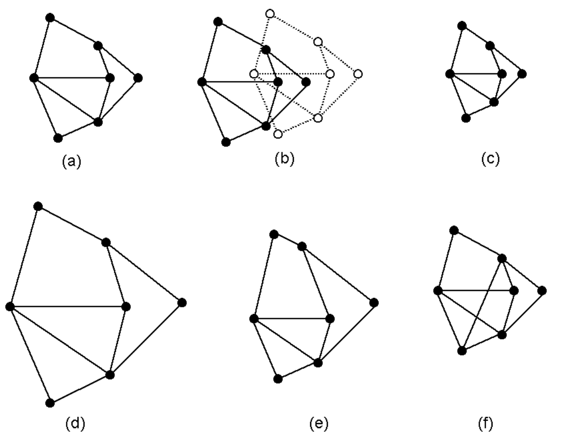

Fig. 7. Parallel point formations.

Example 12. *Consider a planar point formation $\mathbb{F}_p$ with bearing constraints shown in Figure 8. We assume that at least one node knows the global coordinate system and the information about this global coordinate system is passed to the other nodes in the formation. The same point formation drawn with bearing constraints in the global coordinate system is shown in Figure 9. This has a rigidity matrix as shown in Table II. We note that given two points $p_i = (x_i, y_i)$ and $p_j = (x_j, y_j)$, then $(p_i - p_j)^\perp = (y_i - y_j, x_j - x_i)$.*

The generic type of rigidity is defined in the same manner with the case of distances.

Theorem 13. *A formation $\mathbb{F}_p$ is generically parallel rigid in 2-dimensional space if and only if*

$$\text{generic rank } \{R(p)\} = 2n - 3.$$

The graph theoretic test is given with the following theorem:

Theorem 14. *A graph $\mathbb{G} = (\mathcal{V}, \mathcal{B})$ is generically rigid in 2-dimensional space if and only if there is a subset $\mathcal{B}' \subseteq \mathcal{B}$ satisfying the following two conditions: (1) $|\mathcal{B}'| = 2|\mathcal{V}| - 3$, (2) For all $\mathcal{B}'' \subseteq \mathcal{B}', \mathcal{B}'' \neq \emptyset, |\mathcal{B}''| \leq 2|\mathcal{V}(\mathcal{B}'')| - 3$, where $|\mathcal{V}(\mathcal{B}'')|$ is the number of vertices that are end-vertices of the edges in $\mathcal{B}''$.*

A graph is *minimally rigid* if it is rigid and it becomes non-rigid under the removal of any edges from the graph.

A. Global Rigidity for Networks with Bearing Information

Recall from §III that although global rigidity implies rigidity for networks with distance information, the

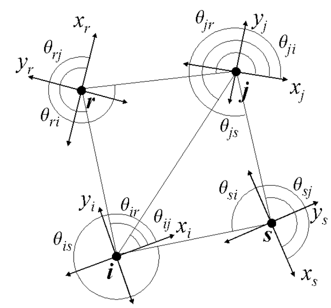

Fig. 8.

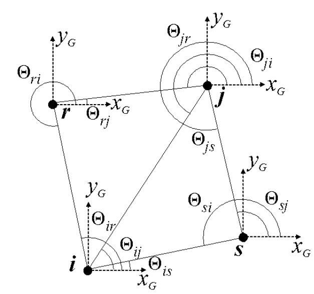

Fig. 9.

reverse is not true. Therefore the conditions for global rigidity is much stronger than rigidity for networks with distance information. On the other hand, as we prove below, for networks with bearing information rigidity also implies global rigidity. We have the following theorem:

Theorem 15. *If $\mathbb{F}_p$ and $\mathbb{F}_q$ are parallel formations in 2-dimensional space, then $\mathbb{F}_p$ is rigid if and only if $\mathbb{F}_p$ is globally rigid under similarity maps.*

Proof: Suppose that $\mathbb{F}_p$ is not globally rigid. Therefore, there is a parallel drawing $\mathbb{F}_q$ which is not similar to $\mathbb{F}_p$ as a configuration. We will show that $\mathbb{F}_p$ is flexible with $\mathbb{F}_q$ as a non-trivial parallel drawing. For all edges $(i,j) \in \mathcal{B}$, $(p_i - p_j)$ is parallel to $(q_i - q_j)$. Therefore, $(p_i - p_j)^\perp \cdot (q_i - q_j) = 0$ as required. Since $\mathbb{F}_p$ is not similar to $\mathbb{F}_q$, there is some pair $(h,k) \notin \mathcal{B}$ such that $p_h - p_k$ is not parallel to $q_h - q_k$. Therefore, $(p_h - p_k)^\perp \cdot (q_h - q_k) \neq 0$. This confirms that $\mathbb{F}_q$ is a non-trivial parallel drawing of $\mathbb{F}_p$.

Conversely, suppose that $\mathbb{F}_p$ is flexible with a non-trivial parallel drawing $\mathbb{F}_q$. Then $\mathbb{F}_q$ itself is the non-similar parallel drawing of $\mathbb{F}_p$ which shows it is not globally rigid. □

1) Sequential Techniques: It is possible to derive useful sufficient conditions and sequential construc-

$R(p)$	i		j		r		s	
(i,j)	$y_i - y_j$	$x_j - x_i$	$y_j - y_i$	$x_i - x_j$	0	0	0	0
(i,r)	$y_i - y_r$	$x_r - x_i$	0	0	$y_r - y_i$	$x_i - x_r$	0	0
(i,s)	$y_i - y_s$	$x_s - x_i$	0	0	0	0	$y_s - y_i$	$x_i - x_s$
(j,r)	0	0	$y_j - y_r$	$x_r - x_j$	$y_r - y_j$	$x_j - x_r$	0	0
(j,s)	0	0	$y_j - y_s$	$x_s - x_j$	0	0	$y_s - y_j$	$x_j - x_s$

TABLE II

RIGIDITY MATRIX EXAMPLE FOR BEARINGS

tions for generically globally rigid networks with bearing information in a similar way that trilaterations are used for generically globally rigid networks with distance information [5]. One operation is the *vertex addition*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we add a new vertex i with two edges between i and two other vertices in $\mathcal{V}$. The other is the *edge splitting*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we remove an edge (j,k) in $\mathcal{L}$ and then we add a new vertex i with three edges by inserting two edges (i,j), (i,k) and one edge between i and one other vertex (other than j,k) in $\mathcal{V}$.

Theorem 16 (vertex addition [18]). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree 2 in 2-dimensional space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ denote the subgraph obtained by deleting i and the edges incident with it. Then $\mathbb{G}$ is minimally parallel rigid if and only if $\mathbb{G}^*$ is minimally parallel rigid.*

Theorem 17 (edge splitting [18]). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree 3; let $\mathcal{V}_i$ be the set of vertices incident to i; and let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ be the subgraph obtained by deleting i and its three incident edges. Then $\mathbb{G}$ is minimally parallel rigid if and only if there is a pair j,k of vertices of $\mathcal{V}_i$ such that the edge (j,k) is not in $\mathcal{L}^*$ and the graph $\mathbb{G}' = (\mathcal{V}^*, \mathcal{L}^* \bigcup (j,k))$ is minimally parallel rigid.*

V. LOCALIZATION IN SUBNETWORKS

In the previous sections, we presented the conditions under which there exist a unique solution for the network localization problem. One might argue that although $\mathbb{G}_{\mathbf{N}}$ fails the conditions for unique localizability, it might still be possible to localize some of the nodes, although not the entire network. Next we consider those cases.

A. Globally rigid subnetworks

Assume that the underlying grounded graph of the network $\mathbf{N}$, namely $\mathbb{G}_{\mathbf{N}} = (\mathcal{V}, \mathcal{E})$, does not satisfy the conditions of Theorem 8. Hence it is true that not all the nodes in $\mathbf{N}$ are localizable. But let us assume that there exists grounded subgraph(s) of $\mathbb{G}_{\mathbf{N}}$, namely $\bar{\mathbb{G}}_1, \bar{\mathbb{G}}_2, \ldots, \bar{\mathbb{G}}_k$ that satisfy the conditions of Theorem 8. Then the subnetworks $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_k$ (with the underlying grounded graphs $\bar{\mathbb{G}}_1, \bar{\mathbb{G}}_2, \ldots, \bar{\mathbb{G}}_k$ respectively) are localizable. That is, all the nodes in $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_k$ are localizable.

In another scenario, globally rigid subnetworks can be merged together to form a larger globally rigid network. In this way, computation can be decentralized. Strategies to merge globally rigid subnetworks were given in [3].

B. Implicitly globally rigid subnetworks

Let us assume that the underlying grounded graph of the network $\mathbf{N}$, namely $\mathbb{G}_{\mathbf{N}}$ is 2-connected, but not 3-connected. Hence it does not satisfy the conditions of Theorem 8. Let us assume that there exists a single cut set denoted by $\mathcal{C} = \{c_1, c_2\}$ where c_1, c_2 are cut vertices. When $c-1, c_2$ are removed then there remains two connected graph components. Let us denote these connected graph components by $\mathbb{K}_1$ and $\mathbb{K}_2$. Let $\hat{\mathbb{G}}_{\mathbf{N}}$ be the grounded graph obtained by inserting the implicit edge (c_1, c_2) to $\mathbb{G}_{\mathbf{N}}$. Let us consider each connected component together with this inserted implicit edge and denote them by $\hat{\mathbb{K}}_1 = \mathbb{K}_1 \cup \{(c_1, c_2)\}$ and $\hat{\mathbb{K}}_2 = \mathbb{K}_2 \cup \{(c_1, c_2)\}$. We have the following proposition:

Proposition 18. *If $\hat{\mathbb{K}}_1$ is rigid and if $\hat{\mathbb{K}}_2$ satisfies the conditions of Theorem 8 then $\hat{\mathbb{K}}_2$ is localizable.*

Proof: Recall that adding an implicit edge does not increase the rank of the rigidity matrix. Given that $\hat{\mathbb{K}}_1$ is rigid, then the edge (c_1, c_2) implicitly exists. Hence we can consider the entire network with this implicit edge inserted, and the rigidity properties of the networks remain the same. After inserting (c_1, c_2), given that $\hat{\mathbb{K}}_2$ satisfies the conditions of Theorem 8 implies that $\hat{\mathbb{K}}_2$ is localizable. □

This result given for one cut set can be generalized to any number of cut sets by applying Proposition 18 repetitively.

VI. Concluding Remarks

In this paper, first we have demonstrated the usefulness of rigidity and parallel drawings for localization in sensor networks. The unique localization of networks from distance and bearing measurements shares a number of features with work in several other active fields of study: rigidity and global rigidity in frameworks; the coordinated formations of autonomous agents; and geometric constraints in CAD. In this paper, we have drawn on techniques and results from these fields, also combined in some previous joint work [5], as well as specific results on global rigidity. With these concepts, we were able to lay a coherent solid foundation for the underlying problem of when a network is uniquely localizable, for almost all configurations of the points. Specifically, we constructed a formation and then a graph for each network such that the localization problem for the network is uniquely solvable, almost always, if and only if the corresponding graph is generically globally rigid. From these connections, we drew specific results on sequential techniques such as trilateration for distances, vertex addition and edge splitting for bearings.

It should be noted that as stated, the localization problem with precise distance and bearing is not in general numerically well posed since even if it is solvable with the given data, it may be unsolvable with data arbitrarily close to that which is given. In practical terms, this means that special attention must be paid to the computation process and to assessing the significance of approximate solutions. It also means that only graphs which are generically globally rigid are capable of having computationally stable solutions for given data sets. This confirms our choice of conceptual framework for this problem. However, we comment that even approximate solutions are hard to compute due to the hardness of the localization problem.

The networks where nodes use both distance and bearing information together will be explored further in a future paper.

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