CUCS-41-83

For Which Error Criteria Can We Solve Nonlinear Equations?

K. Sikorski
Department of Computer Science
Columbia University
New York, New York 10027
(On leave from the University of Warsaw)

H. Woźniakowski

Institute of Informatics
University of Warsaw
Warsaw, Poland

and

Department of Computer Science
Columbia University
New York, New York 10027

January 1983

Abstract

For which error criteria can we solve a nonlinear scalar equation $f(x) = 0$ where f is a real function on the interval $[a,b]$? The information on f consists of n adaptive evaluations of arbitrary linear functionals and an algorithm is any mapping based on these evaluations.

For the root criterion we prove there does not exist an algorithm to find a point x such that $|x-\alpha| \leq \varepsilon$ where α is a zero of f and $\varepsilon < (b-a)/2$. This holds for arbitrary n and for the class of infinitely many times differentiable functions with all simple zeros. We do not assume that $f(a)f(b) \leq 0$.

For the residual criterion we show almost optimal information and algorithm. More precisely, we prove that if x is the value computed by our algorithm then $f(x) = O(n^{-r})$ where r measures the smoothness of the class of functions f.

Finally a general error criterion is introduced and some of our results are generalized.

1. Introduction

A number of error criteria are commonly used in practice for the approximate solution of a nonlinear scalar equation $f(x) = 0$ where $f:[a,b] \to \mathbb{R}$. For instance one may want to find a number x such that one of the following conditions is satisfied:

(1.1) root criterion : $|x-\alpha| \le \varepsilon$,

(1.2) relative root criterion : $|x-\alpha| \le \varepsilon(|\alpha|+\delta)$, $\delta \ge 0$,

(1.3) residual criterion : $|f(x)| \le \varepsilon$,

(1.4) relative residual criterion : $|f(x)| \le \varepsilon|f'(x)|$

where α is a real zero of f and ε is a given nonnegative number.

We study for which error criteria it is possible to find such a number x and, if it is possible, what is an optimal algorithm for finding x.

We assume that f belongs to a class of functions and that we know n adaptive evaluations of arbitrary linear functionals on f. By an algorithm we mean a mapping depending on these n evaluations; see [6].

For the root criterion we prove that there does not exist an algorithm to find x satisfying (1.1) with $\varepsilon < (b-a)/2$ for the class of infinitely many times differentiable functions with simple zeros and whose seminorm is bounded by one. (We do not assume that f has opposite signs at a and b.) Note that this result holds for arbitrary large n and independently of which linear functionals are evaluated. The same result holds for the relative root criterion with $\varepsilon < (b-a)/(b+a+2\delta)$ and $a \geq 0$.

For the residual criterion we deal with the class of functions having zeros and whose (r-1)-st derivative is absolutely continuous and the infinity norm of the r-th derivative is bounded by one, $r \geq 1$. We find almost optimal information and algorithm by the extensive use of the Gelfand n-widths. This information consists of n <u>nonadaptive</u> function evaluations and the algorithm is based on perfect splines interpolating f. This algorithm yields a point x such that $f(x) = O(n^{-r})$.

For small r, we present in Section 4 a different algorithm which is also almost optimal and whose computation is much simpler than the computation of the algorithm based on perfect splines.

If n is large enough, $n = \Theta(\varepsilon^{-1/r})$, then the residual criterion is satisfied. By contrast we prove that the relative residual criterion is never satisfied.

In Section 5 we discuss a general error criteria and find a lower bound on the error of optimal algorithm in terms of the Gelfand width.

2. Root Criterion

Let $C^\infty = C^\infty[a,b]$ be the linear space of infinitely often differentiable functions $f, f:[a,b] \to \mathbb{R}$. Let $S(f)$ denote the set of all zeros of f,

$$(2.1) \qquad S(f) = \{z \in [a,b] : f(z) = 0\}.$$

Let $\|\cdot\|$ be an arbitrary seminorm defined on C^∞. We consider the subclass F of C^∞ consisting of functions which have only simple zeros and whose seminorm is bounded by one, i.e.,

$$(2.2) \qquad F = \{f \in C^\infty : S(f) \neq \emptyset,\ f'(z) \neq 0,\ z \in S(f) \text{ and } \|f\| \leq 1\}.$$

For a given ε, $\varepsilon \geq 0$, we want to find a point z satisfying a root criterion, i.e., such that

$$(2.3) \qquad \mathrm{dist}(z, S(f)) \leq \varepsilon.*$$

To solve this problem we use an <u>adaptive linear</u> information operator N_n which is defined as follows, see [6]. Let $f \in C$ and

---

*For two subsets X and Y of $\mathbb{R}$, by dist(X,Y) we mean $\mathrm{dist}(X,Y) = \inf_{x \in X} \inf_{y \in Y} |x-y|$.

$$N_n(f) = [L_1(f), L_2(f;y_1), \ldots, L_n(f;y_1,\ldots,y_{n-1})] \tag{2.4}$$

where $y_i = L_i(f;y_1,\ldots,y_{i-1})$ and

$$L_{i,f}(\cdot) \overset{\mathrm{df}}{=} L_i(\cdot\,;y_1,\ldots,y_{i-1}) : C^\infty \to \mathbb{R} \tag{2.5}$$

is a linear functional, $i = 1,2,\ldots,n$.

The total number of functional evaluations n is called the cardinality of N_n.

Knowing $N_n(f)$ we approximate a zero of f by an algorithm φ which is a mapping

$$\varphi : N_n(C^\infty) \to [a,b]. \tag{2.6}$$

The error of the algorithm φ is defined as

$$e(\varphi) = \sup_{f \in F} \operatorname{dist}(\varphi(N_n(f)), S(f)). \tag{2.7}$$

Let $\Phi(N_n)$ be the class of all algorithms using information N_n. From [6] and [7] we know that

$$\inf_{\varphi \in \Phi(N_n)} e(\varphi) = r(N_n) \tag{2.8}$$

where $r(N_n)$ is the radius of information. It is easy to show that

$$r(N_n) = \sup\{\operatorname{dist}(S(\tilde{f}), S(\tilde{\tilde{f}}))/2 : f, \tilde{f}, \tilde{\tilde{f}} \in F, N_n(\tilde{f}) = N_n(\tilde{\tilde{f}}) = N_n(f)\}. \tag{2.9}$$

Let Ψ_n be the class of all adaptive linear information operators of the form (2.4). We are ready to prove the following theorem.

Theorem 2.1:

(2.10) $\quad r(N_n) = (b-a)/2, \quad \forall N_n \in \Psi_n.$ □

Proof: Setting $\varphi(N_n(f)) = (a+b)/2$ we get $e(\varphi) \le (b-a)/2$. Thus $r(N_n) \le (b-a)/2$ due to (2.8). To prove the reverse inequality we construct for every γ, $0 < \gamma < (b-a)/2$, two functions $\tilde{f}$ and $\tilde{\tilde{f}}$ from F such that $N_n(\tilde{f}) = N_n(\tilde{\tilde{f}})$ and $\text{dist}(S(\tilde{f}), S(\tilde{\tilde{f}})) \ge b-a-2\gamma$. Then (2.10) will follow from (2.9) with γ tending to zero.

We first construct the function $\tilde{f}$. Define the points

(2.11) $\quad x_i = a + i\gamma/(n+1)$

for $i = 0,1,\ldots,n+1$ and the functions

$$h_i(x) = \begin{cases} \exp(16((n+1)/\gamma)^4 \exp(-1/((x-x_{i-1})^2(x-x_i)^2)) & \text{if } x \in [x_{i-1} x_i], \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1,2,\ldots,n+1$. Note that $h_i \in C^\infty$ and $\max_{x\in[a,b]} |h_i(x)| = 1$.

Next let $d = \max(\|1\|, \max_{1\le i\le n+1}\|h_i\|)$. Take a positive δ such that

$$\delta < 1/(4(n+1)d) \qquad \text{if } d > 0.$$

Let $\delta(x) = \delta$ for $x \in [a,b]$. Applying N_n to the function $\delta(\cdot)$ we get the information operator $N_{n,\delta}$, see (2.5),

$$N_{n,\delta}(f) = [L_{1,\delta}(f),\ldots,L_{n,\delta}(f)].$$

Let $\vec{c} = (c_1,\ldots,c_{n+1})$ be a nonzero solution of the homogeneous system of n linear equations with $n + 1$ unknowns,

$$\Sigma_{i=1}^{n+1} c_i L_{j,\delta}(h_i) = 0, \quad j = 1,2,\ldots,n.$$

Let $|c_k| = \max_{1\le i\le n+1} |c_i|$. Define the function $H \in C^\infty$ as

$$H = \frac{\delta}{|c_k|} \Sigma_{i=1}^{n+1} c_i h_i.$$

Let $c \in (1,3]$. Define the function

$$f_c(x) = \begin{cases} \delta + cH(x) & \text{if } c_k < 0, \\ \delta - cH(x) & \text{if } c_k > 0. \end{cases}$$

Note that $f_c \in C^\infty$. If $d = 0$ then $\|f_c\| = 0$. If $d > 0$ then

$$\|f_c\| \le \delta\|1\| + c\|H\| \le \|1\|/(4(n+1)d) + 3\delta(n+1)d$$

$$\le 1/4 + 3/4 = 1.$$

Observe that $f_c(x_i) = \delta$ and $f_c((x_{k-1}+x_k)/2) = \delta - c\delta < 0$. Thus f_c has a zero. It is easy to see that f_c has at most $2(n+1)$

zeros and $S(f_c) \subset [a,a+\gamma]$. Further, note that $f_c'(x) = 0$ iff $x = x_i$, $x = (x_{i-1}+x_i)/2$, $x \in [x_{j-1},x_j]$ if $c_j = 0$ or $x \in [a+\gamma,b]$. There exists $c = c^* \in (1,3]$ such that $c^*|H((x_{i-1}+x_i)/2)| \neq \delta$ for $i = 1,2,\ldots,n+1$. Therefore the function $\tilde{f} = f_{c^*}$ has only simple zeros and $\tilde{f} \in F$.

To construct $\tilde{\tilde{f}}$ we proceed as above with x_i replaced by $x_i^* = b - i\gamma/(n+1)$, $i = 0,1,\ldots,n+1$. Then $\tilde{\tilde{f}} \in F$ and $S(\tilde{\tilde{f}}) \subset [b-\gamma,b]$. Hence $\text{dist}(S(\tilde{f}),S(\tilde{\tilde{f}})) \geq b-a-2\gamma$. Note that $N_n(\tilde{f}) = N_n(\tilde{\tilde{f}}) = N_n(\delta(\cdot))$ for small δ. This completes the proof. □

Theorem 2.1 states that the error of any algorithm is at least $(b-a)/2$. Thus if $\varepsilon < (b-a)/2$ then there exists no algorithm for which the root criterion is satisfied.

3. Residual Criterion

Let $W_\infty^r[a,b]$ be the space of functions $f:[a,b] \to \mathbb{R}$ whose $(r-1)$-st derivative is absolutely continuous and such that the infinity norm of the r-th derivative is finite, $\|f^{(r)}\|_\infty < +\infty$, $r \geq 1$. Let $W_\infty^r = \{f \in W_\infty^r[a,b]:\|f^{(r)}\|_\infty \leq 1\}$. Recall that $S(f) = \{z \in [a,b]:f(z) = 0\}$. Let

$$(3.1) \qquad F = \{f \in W_\infty^r : S(f) \neq \emptyset\}.$$

For a given $\varepsilon > 0$ we seek a point x for which the residual criterion is satisfied, i.e.,

(3.2) $$|f(x)| \leq \varepsilon.$$

To solve this problem we use adaptive linear information N_n and an algorithm φ using N_n as defined by (2.4) and (2.6) with C^∞ replaced by $W_\infty^r[a,b]$. The error of the algorithm is now defined as

$$e(\varphi) = \sup_{f \in F} |f(\varphi(N_n(f)))|.$$

Then (2.8) holds with the radius of information given by (see also [3] and [7])

(3.3) $$r(N_n) = \sup_{f \in F, x \in [a,b]} \inf \sup\{|\tilde{f}(x)| : \tilde{f} \in F, N_n(\tilde{f}) = N_n(f)\}.$$

Let $C = C[a,b]$ be the space of continuous functions defined on $[a,b]$ and equipped with the norm $\|f\|_C = \max_{x \in [a,b]} |f(x)|$.

By $d^n(W_\infty^r, C)$ we mean the Gelfand n-th width of W_∞^r in the space C, i.e.,

(3.4) $$d^n(W_\infty^r, C) = \inf_{L_1, \ldots, L_n} \sup \{\|f\|_C : f \in W_\infty^r, L_1(f) = \ldots = L_n(f) = 0\}$$

where $L_1, \ldots, L_n$ are linear functionals. It is known, see [5], that

$$d^n(W_\infty^r, C) = \left(\frac{b-a}{2}\right)^r d^n(W_\infty^r, C[-1,1]) = \left(\frac{b-a}{\pi n}\right)^r K_r(1+o(1)),$$

as $n \to \infty$

where K_r is the Favard constant, $K_r \in [1,\pi/2]$.

We first show that the radius $r(N_n)$ of any information operator N_n from Ψ_n is no less than $d^{n+1}(W_\infty^r,C)$.

Theorem 3.1:

$$r(N_n) \geq d^{n+1}(W_\infty^r,C), \qquad N_n \in \Psi_n. \qquad \square$$

Proof: Let φ be any algorithm using N_n. Let $d^{n+1} = d^{n+1}(W_\infty^r,C)$ and take $\eta \in (0,d^{n+1})$. Applying N_n to the function $\delta(\cdot)$,

$$\delta(x) = \begin{cases} d^{n+1} - \eta & \text{if } d^{n+1} < +\infty \\ \eta & \text{otherwise,} \end{cases}$$

we get the information operator $N_{n,\delta}$, $N_{n,\delta}(f) = [L_{1,\delta}(f),\ldots,L_{n,\delta}(f)]$, see (2.5). Let $z = \varphi(N_n(\delta))$. Choose a function f^* from W_∞^r such that $N_{n,\delta}(f^*) = 0$, $f^*(z) = 0$ and

$$\|f^*\|_C \geq \begin{cases} a-\eta & \text{if } a < +\infty \\ \eta & \text{otherwise,} \end{cases}$$

where $a = \sup\{\|f\|_C : f \in W_\infty^r, N_{n,\delta}(f) = 0,\ f(z) = 0\}$. From (3.4) we conclude that

$$\|f^*\|_C \geq \begin{cases} d^{n+1}-\eta & \text{if } d^{n+1} < +\infty \\ \eta & \text{otherwise.} \end{cases}$$

Thus there exists a point $y \in [a,b]$ such that

$$|f^*(y)| \geq \begin{cases} d^{n+1} - \eta & \text{if } d^{n+1} < +\infty \\ \eta & \text{otherwise.} \end{cases}$$

Define

$$g(x) = \begin{cases} d^{n+1} - \eta - \text{sign}(f^*(y)) f^*(x) & \text{if } d^{n+1} < +\infty, \\ \eta - \text{sign}(f^*(y)) f^*(x) & \text{otherwise.} \end{cases}$$

Note that $\|g^{(r)}\| = \|f^{*(r)}\|$, $g(y) \leq 0$ and $g(z) > 0$. Thus $g \in F$. Since $N_n(g) = N_n(\delta)$ then $\varphi(N_n(g)) = z$. By taking the supremum over F we get

$$e(\varphi) \geq |g(z)| = \begin{cases} d^{n+1} - \eta & \text{if } d^{n+1} \leq \infty, \\ \eta & \text{otherwise.} \end{cases}$$

Since η is arbitrary we get $e(\varphi) \geq d^{n+1}$ which completes the proof. □

We now exhibit an infromation operator N_n^*, and an algorithm φ^* using N_n^*, such that $e(\varphi^*) \leq 2d^n(W_\infty^*, C)$.

Following [2], [5] pp. 130-135, 261-263 and [6] p. 129 assume that $n \geq r$ and define $X_{n-r,r}$ as the class of perfect splines $s:[a,b] \to \mathbb{R}$ of degree r which have $n - r$ knots, i.e., for every s from $X_{n-r,r}$ there exists $t_i = t_i(s)$, $a \leq t_1 \leq \ldots \leq t_{n-r} \leq b$ and $a_i = a_i(s)$ such that

$$s(t) = \frac{(t-a)^r}{r!} + \Sigma_{i=1}^r a_i t^{i-1} + \frac{2}{r!} \Sigma_{i=1}^{n-r} (-1)^i (t-t_i)_+^r.$$

There exists a unique (up to multiplication by -1) perfect spline $s_{n-r,r}$ from $X_{n-r,r}$ with the minimal norm, i.e.,

$$\|s_{n-r,r}\|_C = \inf_{s \in X_{n-r,r}} \|s\|_C.$$

The spline $s_{n-r,r}$ has n distinct zeros $x_1^*,\ldots,x_n^*$ and

$$\|s_{n-r,r}\|_C = d^n(W_\infty^r, C).$$

Define the information operator

$$N_n^*(f) = [f(x^*_1),\ldots,f(x_n^*)], \quad f \in W_\infty^r.$$

We now define the algorithm φ^* using N_n^* as follows. Let u and v be perfect splines of degree r with n-r knots η_i and ξ_i respectively, $i = 1,2,\ldots,n-r$, interpolating f at x_i^*, i.e., $u(x_i^*) = v(x_i^*) = f(x_i^*)$, and such that

$$u^{(r)}(x) = (-1)^i \quad \text{for} \quad \eta_i < x < \eta_{i+1}, \ i = 0,1,\ldots,n-r,$$

where $\eta_0 = x_1^*$, $\eta_{n-r+1} = x_n^*$,

$$v^{(r)}(x) = (-1)^{i+1} \quad \text{for} \quad \xi_i < x < \xi_{i+1}, \ i = 0,1,\ldots,n-r,$$

where $\xi_0 = x_1^*$ and $\xi_{n-r+1} = x_n^*$. Define

$$f^-(x) = \min(u(x),v(x)),$$

$$f^{+}(x) = \max(u(x),v(x)).$$

It is shown in [1] that f^{-} and f^{+} are the envelopes for the family of functions from W_{∞}^{r} having the same information as f, i.e.,

$$f^{-}(x) \le \tilde{f}(x) \le f^{+}(x), \quad x \in [a,b],$$

where $\tilde{f} \in W_{\infty}^{r}$ and $N_n(\tilde{f}) = N_n(f)$.

Let $f^* = (f^{+} + f^{-})/2$ and let z^* satisfy the equation $|f^*(z^*)| = \min_{z\in[a,b]} |f^*(z)|$. Then the algorithm φ^* is defined as

$$\varphi^*(N_n^*(f)) = z^*.$$

We now prove

<u>Theorem 3.2</u>:

$$e(\varphi^*) \le 2d^{n}(W_{\infty}^{r},C).$$ □

<u>Proof</u>: Let $f \in F$ and z be a zero of f. It is known (see [2] and [6]) that $\|f^*-f\|_C \le d^n = d^n(W_{\infty}^{r},C)$ for every f. Therefore

$$|f^*(z^*)| \le |f^*(z)| = |f^*(z)-f(z)| \le \|f^*-f\|_C \le d^n$$

and

$$|f(z^*)| \le |f^*(z^*)-f(z^*)| + |f^*(z^*)| \le 2d^n.$$

The proof is completed by taking the supremum over F. □

From Theorems 3.1 and 3.2 we have the following corollary.

Corollary 3.1: The information N_n^* and the algorithm φ^* are almost optimal, i.e.,

$$r(N_n^*) = c_n(1+o(1)) \inf_{N_n \in \Psi_n} r(N_n) = \left(\frac{b-a}{\pi n}\right)^r K_r(1+o(1)), \quad \text{as } n \to \infty,$$

and

$$e(\varphi^*) = c_n' r(N_n^*)(1+o(1)), \quad \text{as } n \to \infty,$$

for some c_n and c_n' from $[1,2]$. □

To guarantee that the residual criterion is satisfied with $x = \varphi^*(N_n^*(f))$ it is enough to define n such that $e(\varphi^*) \le \varepsilon$. Due to Corollary 3.1 we have

$$n = n(\varepsilon) = \frac{b-a}{\pi}\varepsilon^{-1/r} \sqrt[r]{K_r c_n' c_n}\,(1+o(1)).$$

Furthermore this n is almost the minimal one for which the residual criterion is satisfied.

4. Algorithm with small combinatory cost.

The almost optimal algorithm φ^* from Section 3 is, in general, nonlinear since the computation of φ^* requires the solution of two nonlinear systems of size n - r (see [1] and [6]). Therefore its combinatory cost may be large. In this section we define the information N_n^{**} and the algorithm φ^{**} which are almost optimal and easy to compute.

Let $n = k\cdot r$ where k is a nonnegative integer. Let $h = (b-a)/k$ and $[a_i,b_i] = [a+(i-1)h, a+ih]$ for $i = 1,2,\ldots,k$. Let

$$g_i(x) = \frac{a_i+b_i}{2} - \frac{a_i-b_i}{2}x$$

be the linear transformation of [-1,1] on $[a_i,b_i]$. Denote $x_{i,j} = g_i(z_j)$ where $z_j = \cos((2j-1)\pi/(2r))$, $j = 1,\ldots,r$, are the zeros of Chebyshev polynomial T_r.

Let F be defined by (3.1). For $f \in F$ define the information N_n^{**} as

$$(4.1) \qquad N_n^{**}(f) = [f(x_{1,1}),\ldots,f(x_{1,r}),\ldots,f(x_{k,1}),\ldots,f(x_{k,r})],$$

and the interpolatory polynomials w_i of degree r-1 satisfying

$$(4.2) \qquad w_i(x_{i,j}) = f(x_{i,j}),\ j = 1,2,\ldots,r.$$

We know that

$$(4.3) \qquad \sup_{x\in[a_i,b_i]} |w_i(x)-f(x)| \le \frac{1}{r!}\left(\frac{b-a}{2k}\right)^r \frac{1}{2^{r-1}} = \left(\frac{b-a}{n}\right)^r \frac{r^r}{r!\,2^{2r-1}}, \quad \forall i.$$

Note that

$$A = \frac{r^r}{r!\,2^{2r-1}}\left(\frac{b-a}{n}\right)^r = \sqrt{\frac{2}{\pi r}}\left(\frac{e}{4}\right)^r\left(\frac{b-a}{n}\right)^r(1+o(1)) \text{ as } r \to \infty.$$

Define the algorithm φ^{**} as

$$(4.4) \qquad \varphi^{**}(N_n^{**}(f)) = x^{**}$$

where x^{**} is chosen from $[a,b]$ such that $\min_{1\le i\le k} |w_i(x^{**})| \le A$. Note that such a point exists. Indeed, since f has a zero α in some subinterval $[a_j,b_j]$, then (4.3) yields

$$(4.5) \qquad \min_{1\le i\le k} \; \min_{x\in[a_i,b_i]} |w_i(x)| \le |w_j(\alpha)| \le A.$$

Inequality (4.3) yields

$$|f(x^{**})| \le 2A$$

and therefore $e(\varphi^{**}) \le 2A$. From this we have the following corollary.

Corollary 4.1: The information N_n^{**} and the algorithm φ^{**} are almost optimal since

$$r(N_n^{**}) = c_n \inf_{N_n\in\Psi_n} r(N_n)$$

and

$$e(\varphi^{**}) = c_n' r(N_n^{**})$$

where

$$c_n, c_n' \in [1,B],$$

for $B = (\pi r)^r/(r!K_r)4^{1-r}(1+o(1))$ as $n \to \infty$. □

Note that for large r we have

$$B = 2\sqrt{\frac{2}{\pi r}}\left(\frac{\pi e}{4}\right)^r(1+o(1)).$$

For small r, $r \leq 4$ say, it is easy to implement (4.4). For instance we may compute $f(x_{1,1}),\ldots,f(x_{1,r})$ and check if $\min_{1\leq j\leq r}|f(x_{1,j})| \leq A$. If so we are done. If not we construct w_1 and compute a point x_1 such that $|w_1(x_1)| = \min_{x\in[a_1,b_1]}|w_1(x)|$. If $|w_1(x_1)| \leq A$ then we are done, if not we compute the next values of f at $x_{2,1},\ldots,x_{2,r}$ and repeat the above procedure. As in (5.5) there exists a point $x_i \in [a_i,b_i]$ such that $|w_i(x_i)| \leq A$ for some i where x_i is defined by $|w_i(x_i)| = \min_{x\in[a_i,b_i]}|w_i(x)|$.

5. General Error Criterion

One may want to solve a nonlinear equation using an error criterion different than (1.1) or (1.3). This can be done as follows.

Let F be a given subclass of functions from a linear space G, and let

$$E: G \times [a,b] \to \mathbb{R}_+. \tag{5.1}$$

For a given $\varepsilon \in \mathbb{R}_+$ and any function f from F we want to find a point $x = x(f,\varepsilon)$ such that

$$(5.2) \qquad E(f,x) \le \varepsilon .$$

We call (5.2) a general error criterion. The examples of the general error criterion are as follows

$$(5.3) \qquad E(f,x) = \inf\{|x-\alpha| : \alpha \in S(f)\}$$

corresponds to the root criterion (1.1),

$$(5.4) \qquad E(f,x) = \inf\{|x-\alpha|/(|\alpha| + \delta) : \alpha \in S(f)\}$$

corresponds to the relative root criterion (1.2),

$$(5.5) \qquad E(f,x) = |f(x)|$$

corresponds to the residual criterion and

$$(5.6) \qquad E(f,x) = \begin{cases} |f(x)/f'(x)| & \text{if } f'(x) \neq 0, \\ +\infty & \text{if } f(x) \neq 0 \text{ and } f'(x) = 0, \\ 0 & \text{if } f(x) = 0 \text{ and } f'(x) = 0 \end{cases}$$

corresponds to the relative residual criterion. To find x satisfying (5.2) we use an information operator N_n and algorithm φ using N_n which are defined as in (2.4) and (2.6). By the error of the algorithm φ we now mean

$$e(\varphi) = \sup_{f \in F} E(f, \varphi(N_n(f))).$$

Thus $x = \varphi(N_n(f))$ satisfies (5.2) for any $f \in F$ iff $e(\varphi) \le \varepsilon$.

It is easy to generalize (2.9) and (3.3) by showing that

$$(5.7)\qquad \inf_{\varphi\in\Phi(N_n)} e(\varphi) = r(N_n)$$

$$= \sup_{f\in F}\ \inf_{c\in[a,b]}\ \sup\{E(\tilde{f},c):\tilde{f}\in F, N_n(\tilde{f}) = N_n(f)\}.$$

We illustrate (5.7) by an example.

<u>Example 5.1</u>: Let F be defined by (2.2) and E by (5.4). Assume for simplicity that $a \geq 0$. In the proof of Theorem 2.1 we used two functions with the same information whose zeros are arbitrarily close to the endpoints of [a,b]. From this we conclude that

$$r(N_n) \geq \inf_{c\in[a,b]} \max\{\frac{|c-a|}{a+\delta},\frac{|c-b|}{b+\delta}\} = \frac{b-a}{b+a+2\delta}.$$

Further note that $\varphi(N_n(f)) = c^* = (2ab+\delta(a+b))/(a+b+2\delta)$ has the error

$$e(\varphi) = \sup_{c\in[a,b]} |c-c^*|/(c+\delta) = \max(\frac{|a-c^*|}{a+\delta},\frac{|b-c^*|}{b+\delta})$$

$$= (b-a)/(a+b+2\delta).$$

Due to (5.7) we have

$$(5.8)\qquad r(N_n) = e(\varphi) = \frac{b-a}{a+b+2\delta}.$$

Note that for $\delta = 0$, $\varphi(N_n(f))$ is the harmonic mean of a and b. Since (5.8) holds for any information operator N_n we conclude that if $\epsilon < (b-a)/(a+b+2\delta)$ then there exists no algorithm for which the relative root criterion is satisfied. □

We now assume a special form of the operator E. Let F be defined by (3.1), $G = W_\infty^r(a,b]$, and let

$$(5.9) \qquad A(f,x) = [L_1(f,x),\ldots,L_k(f,x)]$$

where $L_i(\cdot,x):G \to \mathbb{R}$ is a linear functional, $i = 1,2,\ldots,k$. Assume that E is of the form

$$(5.10) \qquad E(f,x) = E(A(f,x),x),$$

i.e., the dependence on f is through $A(f,x)$. Let $d^{n+k+1} = d^{n+k+1}(W_\infty^r,C)$ by the Gelfand $(n+k+1)$-st width, see Section 3. We generalize Theorem 3.1 by proving

<u>Theorem 5.1</u>: Let E be s-homogeneous, i.e.,

$$E(A(cf,x),x) = c^s E(A(f,x),x)$$

for all $(c,f,x) \in \mathbb{R} \times G \times [a,b]$. Then

$$(5.11) \qquad r(N_n) \geq (d^{n+k+1})^s \inf_{z\in[a,b]} E(A(1,z),z).$$ □

<u>Proof:</u> We sketch the proof since it is similar to the proof of Theorem 3.1. Let $\eta \in (0,d^{n+k+1})$. Apply N_n to the function

$\delta(x) = d^{n+k+1} - \eta$ getting $N_{n,\delta}$. Let $z = \varphi(N_n(\delta))$ for an algorithm φ. Choose f^* from W_∞^r such that $N_{n,\delta}(f^*) = 0$, $A(f^*,z) = 0$, $f^*(z) = 0$ and

$$\|f^*\|_C + \eta \geq \sup\{\|f\|_C : f \in W_\infty^r : N_{n,\delta}(f)=0, A(f,z)=0, f(z)=0\}.$$

Then $|f^*(y)| = \|f^*\|_C \geq d^{n+k+1} - \eta$ for some y from $[a,b]$. The function $g(x) = d^{n+k+1} - \eta - \operatorname{sign}(f^*(y))f^*(x)$ belongs to F, $\varphi(N_n(g)) = z$ and $e(\varphi) \geq E(A(d^{n+k+1}-\eta z),z)$ $= (d^{n+k+1}-\eta)^s E(A(1,z),z)$. Since φ and η are arbitrary, (5.11) is proven. □

We illustrate Theorem 5.1 by two examples. Consider the relative residual criterion, i.e., E is given by (5.6) and $A(f,x) = [f(x), f'(x)]$. Then $s = 0$ and $E(A(1,z),z) = +\infty$, $\forall z$. Thus (5.11) yields $r(N_n) = +\infty$, $\forall N_n$. This means that there exists no algorithm for which the relative residual criterion is satisfied no matter how large ε.

As the second example consider $A(f,x) = f(x)$ and

$$E(f,x) = |f(x)|^s.$$

Then E is s-homogeneous and (5.11) holds with $K = 1$ and $E(A(1,z),z) \equiv 1$. Using Theorem 3.2 it is easy to verify that there exists an information operator N_n such that $r(N_n) \leq 2^s (d^n)^s$.

This shows that (5.11) is essentially sharp for this case.

6. Final Remark

We stress that in this paper we <u>do not</u> assume that a function f from the class F has opposite signs at the endpoints of the interval. If we shrink the class F to the subclass F_1, defined as $F_1 = \{f \in F: f(a) \leq 0, f(b) \geq 0$ and f has one zero which is simple$\}$ then the results of the paper for the root criterion do not hold. It turns out, see [4], that the bisection algorithm and the bisection information are optimal in this case, and the error is $(b-a)/2^{n+1}$. This shows that the assumption of different signs at the endpoints carries much more information than the smoothness of f.

Acknowledgements

We are greatly indebted to G.W. Wasilkowski who suggested to analyze general error criteria and outlined the approach to Theorem 5.1. We wish to thank J.F. Traub for his valuable comments and remarks.

References

[1] Gaffney, P.W., and Powell, M.J.D., Optimal interpolation, in Numerical Analysis (G.A. Watson ed.) Lecture Notes in Math., Vol 506, pp. 90-100, Springer Verlag, Berlin and New York, 1976.

[2] Micchelli, C.A., Rivlin, T.F., and Winograd, S., The Optimal Recovery of Smooth Functions, Numer. Math. 26 (1976), 191-200.

[3] Sikorski, K., Optimal Solution of Nonlinear Equations Satisfying a Lipschitz Condition, Dept. of Computer Science Report, Columbia University, 1982.

[4] Sikorski, K., Bisection is Optimal, Numer Math., 1982.

[5] Tikhomirov, V.M., Some Problems of Approximation Theory, Moscow State Univ., Moscow, 1976, (in Russian).

[6] Traub, J.F., and Wozniakowski, H., A General Theory of Optimal Algorithms, Academic Press, New York, 1980.

[7] Traub, J.F., Wasilkowski, G.W., and Wozniakowski, H., Information, Uncertainty, Complexity, Addison-Wesley, Reading, Mass., 1983.