On decision trees, influences, and learning monotone decision trees

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Abstract

In this note we prove that a monotone boolean function computable by a decision tree of size s has average sensitivity at most $\sqrt{\log_2 s}$. As a consequence we show that monotone functions are learnable to constant accuracy under the uniform distribution in time polynomial in their decision tree size.

1 Decision trees

Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a boolean function.

Fourier notions: Throughout this paper we view $\{-1,1\}^n$ as a probability space under the uniform distribution. Recall f's *Fourier expansion*,

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S)\chi_S(x),$$

where $\chi_S(x) = \prod_{i \in S} x_i$ and $\hat{f}(S) = \mathbf{E}_x[f(x)\chi_S(x)]$. We also recall the notions of influence and average sensitivity: The *influence of i on f* is $\mathrm{Inf}_i(f) = \Pr_x[f(x) \neq f(x^{(i)})]$, where $x^{(i)}$ denotes x with the ith bit flipped; if f is a monotone function then $\mathrm{Inf}_i(f) = \hat{f}(\{i\})$. We shall henceforth write $\hat{f}(i)$ in place of $\hat{f}(\{i\})$. The *average sensitivity of f* is $\mathrm{I}(f) = \sum_{i=1}^n \mathrm{Inf}_i(f)$.

Decision trees: Suppose we have a decision tree computing $f\colon \{-1,1\}^n \to \{-1,1\}$; we will always assume (without loss of generality) that no variable appears more than once on any path of the tree. Note that picking a uniformly random input $x \in \{-1,1\}^n$ is equivalent to the following two-step procedure: First, pick a uniformly random path P in the tree by starting at the root and assigning to the variables encountered uniformly at random until a leaf is

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reached. Second, assign uniformly at random to those variables as yet unset. Corresponding to the first step of this process we define a collection of random variables $P_1, \ldots, P_n$ as follows:

$$P_i = \begin{cases} 1 & \text{if the variable } i \text{ is encountered on the random path and } x_i \text{ is chosen to be } 1, \\ -1 & \text{if the variable } i \text{ is encountered on the random path and } x_i \text{ is chosen to be } -1, \\ 0 & \text{if the variable } i \text{ is not encountered on the random path.} \end{cases}$$

For each i we have $\mathbf{E}[P_i] = 0$; a slight amount of reflection reveals that also $\mathbf{E}[P_i \mid P_j] = 0$ for all $i \neq j$. Hence while P_i and P_j are not independent we do have $\mathbf{E}[P_i P_j] = 0$ for all $i \neq j$. We write ΣP for $\sum_{i=1}^n P_i$, the sum of the bit assignments made along the random path P, and we also write $\mathrm{len}(P)$ for the length of the random path P; another way of expressing $\mathrm{len}(P)$ is $\sum_{i=1}^n (P_i)^2$.

Consider the two-step procedure for choosing $x \in \{-1,1\}^n$ at random: first choose P at random, assigning randomly to the variables on the path; then choose the remaining unset variables uniformly at random. Since the value of $f(x)$ is fixed after the first step in the procedure, we may denote this value by $f(P)$.

Proposition 1 *Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be computed by a decision tree with paths P. Then*

$$\sum_{i=1}^n \hat{f}(i) = \mathop{\mathbf{E}}_P[f(P) \cdot \Sigma P].$$

Proof:

$$\begin{aligned} \sum_{i=1}^n \hat{f}(i) &= \sum_{i=1}^n \mathop{\mathbf{E}}_{x \in \{-1,1\}^n} [f(x) x_i] \\ &= \mathop{\mathbf{E}}_{x \in \{-1,1\}^n} \left[f(x) \sum_{i=1}^n x_i \right] \\ &= \mathop{\mathbf{E}}_{P;\ x_j:\ P_j = 0} \left[f(P) \left(\sum_{i:\ P_i \neq 0} x_i + \sum_{j:\ P_j = 0} x_j \right) \right] \\ &= \mathop{\mathbf{E}}_P \Big[f(P) \Big(\Sigma P + \mathop{\mathbf{E}}_{x_j:\ P_j = 0} \Big[\sum_{j:\ P_j = 0} x_j \Big] \Big) \Big] \\ &= \mathop{\mathbf{E}}_P [f(P) \cdot \Sigma P]. \end{aligned}$$

The final equality holds since $\mathbf{E}[x_j | P_j = 0] = 0$ for each j. □

Theorem 1 *Let $f\colon \{-1,1\}^n \to \{-1,1\}$.*

1. *If f is computed by a decision tree of size[1] s then $\sum_{i=1}^n \hat{f}(i) \leq \sqrt{\log_2 s}$.*

2. *If f is computed by a decision tree of depth d then $\sum_{i=1}^n \hat{f}(i) \leq \mathrm{I}(\mathsf{Maj}_d) \sim \sqrt{\frac{2}{\pi}} \sqrt{d} \leq \sqrt{d}$.*

If f is monotone we can replace $\sum_{i=1}^n \hat{f}(i)$ by $\mathrm{I}(f)$.

Proof: Since f is ± 1-valued, from the Proposition it is clear that

$$\sum_{i=1}^n \hat{f}(i) \leq \mathop{\mathbf{E}}_P[|\Sigma P|]$$

[1] Number of leaves.

with equality iff f computes the majority of the bits along each of its decision tree paths — i.e., $\mathrm{sgn}(\Sigma P)$ (f may output anything if the bits split evenly.)

In case (1), we proceed as follows:

$$\mathop{\mathbf{E}}_{P}[|\Sigma P|] \le \sqrt{\mathop{\mathbf{E}}_{P}[|\Sigma P|^2]} = \sqrt{\mathop{\mathbf{E}}_{P}\Big[\sum_{i,j=1}^{n} P_i P_j\Big]} = \sqrt{\mathop{\mathbf{E}}_{P}[\mathrm{len}(P)] + \sum_{i\neq j}\mathop{\mathbf{E}}_{P}[P_i P_j]} = \sqrt{\mathop{\mathbf{E}}_{P}[\mathrm{len}(P)]}.$$

(At this point we have proved the upper bound of $\sqrt{d}$ in case (2).) It remains to show that $\mathbf{E}_P[\mathrm{len}(P)] \le \log_2 s$; we use induction on s. The result is obvious when $s = 2$; for larger s, suppose we have a size-s tree in which the left subtree of the root has size s_1 and the right subtree of the root has size s_2, with $s = s_1 + s_2$. The expected length of a random path in such a tree is 1 plus half the expected length in the left subtree plus half the expected length in the right subtree. By induction this is at most $1 + \frac{1}{2}\log_2 s_1 + \frac{1}{2}\log_2 s_2 = \log_2(2\sqrt{s_1 s_2}) \le \log_2(s_1 + s_2) = \log_2 s$ where we have used the AM-GM inequality.

In case (2), we instead note in upper-bounding $\mathbf{E}_P[|\Sigma P|]$ it doesn't hurt to assume that the tree is a full depth-d tree; this is because if we have a path of depth less than d, we can extend it redundantly, querying an irrelevant variable — if ΣP for the path was nonzero then $\mathbf{E}_P[|\Sigma P|]$ is unchanged, if ΣP for the path was zero then $\mathbf{E}_P[|\Sigma P|]$ will increase. Now note that $\mathbf{E}_P[|\Sigma P|]$ does not depend on the names of the variables labeling the nodes of the tree; hence we may assume that all nodes at level ℓ read x_ℓ, for $\ell = 1 \dots d$. But now equality in $\sum_{i=1}^{n} \hat{f}(i) \le \mathbf{E}_P[|\Sigma P|]$ occurs if the decision tree computes Maj_d, as claimed. The asymptotic formula $\mathrm{I}(\mathsf{Maj}_d) = (1 + o(1))\sqrt{\frac{2}{\pi}}\sqrt{d}$ is well known. □

Remarks:

1. In the case of monotone functions with depth-d decision trees, Theorem 1 generalizes the well-known edge-isoperimetric inequality on the discrete n-cube, which says that for monotone $f\colon \{-1,1\}^n \to \{-1,1\}$, $\mathrm{I}(f) \le \mathrm{I}(\mathsf{Maj}_n)$.
2. We conjecture that if $s = s(d)$ is the minimal size of a decision tree computing Maj_d, then every function f computable by a decision tree of size s has $\sum_{i=1}^{n} \hat{f}(i) \le \sum_{i=1}^{n} \widehat{\mathsf{Maj}_d}(i)$.

2 Learning monotone decision trees

The following result is immediate from inspecting the proof of Friedgut's '98 theorem about functions with low average sensitivity [Fri98]:

Theorem 2 *Let $f\colon \{-1,1\}^n \to \{-1,1\}$, $\epsilon > 0$, $t = 2\mathrm{I}(f)/\epsilon$, and $J = \{i : \mathrm{Inf}_i(f) \ge t3^{-t}\}$. Then $|J| \le 3^t$, and furthermore*

$$\sum_{S\colon\ S\subseteq J,\ |S|\le t} \hat{f}(S)^2 \ge 1 - \epsilon.$$

Combining Theorems 1 and 2 with the idea behind the monotone DNF learning algorithm of [Ser01], we get the following uniform distribution algorithm for learning monotone functions in time polynomial in their decision tree size:

Theorem 3 *The class of monotone functions can be learned under the uniform distribution to accuracy ϵ (with confidence $1 - \delta$) in time $s^{O(1/\epsilon^2)} \cdot \mathrm{poly}(n) \cdot \log(1/\delta)$, where s represents decision tree size.*

Proof: Let f be the unknown monotone function to be learned. We may assume that the algorithm knows s, the decision tree size of f, by a standard doubling argument. From Theorem 1 we know that $\mathrm{I}(f) \leq \sqrt{\log_2 s}$; let $t = 2\sqrt{\log_2 s}/\epsilon$. Since f is monotone, its influences are equal to its degree-one Fourier coefficients and thus can be accurately estimated from uniformly random samples. The algorithm first determines the set $J = \{i : \mathrm{Inf}_i(f) \geq t3^{-t}\}$. It then estimates all Fourier coefficients $\hat{f}(S)$ such that $|S| \leq t$ and $S \subseteq J$. By Theorem 2 this gives the algorithm all but ϵ of f's spectrum; it is well known that this is sufficient for learning f to accuracy ϵ. To conclude we note that up to the $\mathrm{poly}(n) \cdot \log(1/\delta)$ the running time is dominated by the number of Fourier coefficients estimated, which is at most $|J|^t = 3^{t^2} = s^{O(1/\epsilon^2)}$. $\square$

References

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