Complexity and tractability of the heat equation

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Abstract

In a previous paper, we developed a general framework for establishing tractability and strong tractability for quasilinear multivariate problems in the worst case setting. One important example of such a problem is the solution of the heat equation $u_t = \Delta u - qu$ in $I^d \times (0, T)$, where I is the unit interval and T is a maximum time value. This problem is to be solved subject to homogeneous Dirichlet boundary conditions, along with the initial conditions $u(\cdot, 0) = f$ over I^d. The solution u depends linearly on f, but nonlinearly on q. Here, both f and q are d-variate functions from a reproducing kernel Hilbert space with finite-order weights of order ω. This means that, although d can be arbitrary large, f and q can be decomposed as sums of functions of at most ω variables, with ω independent of d.

In this paper, we apply our previous general results to the heat equation. We study both the absolute and normalized error criteria. For either error criterion, we show that the problem is *tractable*. That is, the number of evaluations of f and q needed to obtain an ε-approximation is polynomial in ε^{-1} and d, with the degree of the polynomial depending linearly on ω. In addition, we want to know when the problem is *strongly tractable*, meaning that the dependence is polynomial only in ε^{-1}, independently of d. We show that if the sum of the weights defining the weighted reproducing kernel Hilbert space is uniformly bounded in d and the integral of the univariate kernel is positive, then the heat equation is strongly tractable.

1 Introduction

Many important multidimensional problems are intractable, i.e., their complexity grows exponentially with their dimension. This often happens when our problem elements come from classical spaces (such as Sobolev or Hölder spaces) and we are using the worst case setting. A great amount of attention has been paid to the problem of rendering these problems tractable (i.e., for finding polynomial-time algorithms) in the worst case setting. For further discussion, see e.g. [5] and [10, Chapter 3].

If we are to vanquish this curse of dimension, we must use different spaces of problem elements. One fruitful idea has been to use a weighted reproducing kernel Hilbert spaces (RKHS) as the source of problem elements. Here, the weights reflect the idea that some variables may be more important than others. Once again, see [5] for a survey on weighted RKHSs.

In particular, a great deal of attention has been paid to weighted RKHSs with finite-order weights. The main idea here is that although we want to solve problems of very high dimension d, the problem elements are often sums of functions that depend on at most ω variables, where ω is independent of d. As an example, in quantum mechanics, one commonly encounters sums

$$q(\mathbf{x}_1,\ldots,\mathbf{x}_{d/3}) = \sum_{1\le i<j\le d/3} \frac{1}{(\|\mathbf{x}_i-\mathbf{x}_j\|^2_{\ell_2(\mathbb{R}^3)}+\alpha^2)^{1/2}}$$

of modified[1] Coulomb pair potentials, see, e.g., [3, pg. 71]. Here, each $\mathbf{x}_i$ belongs to $\mathbb{R}^3$, so that q depends on d scalar variables; however, each term of q only depends on 6 variables. Hence, $\omega = 6$ for this example.

Finite-order weighted RKHSs were first studied in [1], which dealt with multivariate integration. They were studied for general multivariate linear problems in [11, 12]. The approach of these latter papers would seem to cover the solution of a linear differential or integral equation $\mathscr{L}u = f$. However, such problems tend to have hidden nonlinearities lurking underneath, since the linear operator $\mathscr{L}$ is often of the form $\mathscr{L}_q$ for some function q. For example, q could be a coefficient appearing in a differential operator, or the kernel function of an integral operator. If u is the solution of the problem $\mathscr{L}_q u = f$, then the mapping $f \mapsto u$ is linear for each q, but the mapping $(f,q) \mapsto u$ is nonlinear.

These considerations have lead us to consider the approximate solution of problems given by an operator S_d, in which the mapping $S_d(\cdot, q)$ is linear for each q. Under mild smoothness conditions, we say that such problems are *quasilinear*. A general framework for investigating the tractability of quasilinear problems using finite-order weighted RKHSs was developed in [15]. This framework was used in [16] to study the tractability of the Helmholtz equation $-\Delta u + qu = f$ on the d-dimensional unit cube I^d. In this paper, we shall use the general framework of [15] to study the tractability of the heat equation.

Let I denote the unit interval and let d be an arbitrary positive integer. For a given non-negative function q on I^d, let

$$\mathscr{L}_q = -\Delta + q,$$

with Δ denoting the d-dimensional Laplacian. We are interested in approximating the solution $u = S_d(f,q)$ of the parabolic partial differential equation

$$\frac{\partial u}{\partial t}(\mathbf{x},t) = -(\mathscr{L}_q u)(\mathbf{x},t) \qquad \forall\, \mathbf{x}\in I^d,\ t\in(0,T).$$

This is a heat equation, with q being the heat transfer rate for conductive loss to the ambient environment. The error of an approximation is given by the maximum value of the $L_2(I^d)$-error at time t, over all $t \in [0,T]$.

Let $F_d \subset L_2(I^d)\times Q_d$ be the set of *problem elements* (f,q) for which we wish to solve the heat equation, where Q_d denotes the non-negative functions in $L_\infty(I^d)$. We study two error criteria:

1. The *absolute error criterion*: Here, we want to guarantee that the worst case error of an algorithm is at most ε.

2. The *normalized error criterion*: Here, we want to guarantee that the worst case error is at most ε times the initial error. (By the *initial error*, we mean the minimal error we can attain without sampling $(f,q)\in F_d$, rather than the error at time $t=0$.)

In addition, we assume that we can compute either arbitrary linear functionals of f and q (continuous linear information Λ^{all}) or function values of f and q (standard information Λ^{std}) for any $(f,q)\in F_d$.

[1] The modification is the inclusion of the positive term α. Physicists often include a small α as a regularization parameter, to make q smooth.

Let $\operatorname{card}(\varepsilon, S_d, F_d, \Lambda)$ denote the minimal number of Λ-evaluations needed to compute an ε-approximation in the worst case setting under the absolute or normalized error criterion. We say that the problem $S = \{S_d\}_{d=1}^{\infty}$ is *tractable* if there exist $C > 0$, $p_{\text{err}} \geq 0$, and $p_{\text{dim}} \geq 0$ such that

$$\operatorname{card}(\varepsilon, S_d, F_d, \Lambda) \leq C \left(\frac{1}{\varepsilon}\right)^{p_{\text{err}}} d^{p_{\text{dim}}} \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{Z}^{++}.$$

If no such p_{err} and p_{dim} exist, then the problem S is said to be *intractable*. Furthermore, the problem S is said to be *strongly tractable* if there exist $C > 0$ and $p_{\text{strong}} > 0$ such that

$$\operatorname{card}(\varepsilon, S_d, F_d, \Lambda) \leq C \left(\frac{1}{\varepsilon}\right)^{p_{\text{strong}}}.$$

Our first task is to briefly look at the case where the first component of F_d is the unit ball of a standard Sobolev space $H^r(I^d)$, with fixed r. We easily find that the heat equation is intractable.

Having shown that our problem is intractable for standard Sobolev spaces, we move on to the case of a weighted RKHS $H(K_d)$. Here the reproducing kernel K_d has the form

$$K_d(\mathbf{x}, \mathbf{y}) = \sum_{\substack{\mathfrak{u} \in \{1,\dots,d\} \\ |\mathfrak{u}| \leq \omega}} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} K(x_j, y_j),$$

where K is the reproducing kernel of a Hilbert space $H(K)$ of univariate functions, and $\gamma_{d,\mathfrak{u}}$ are non-negative numbers (weights). The first component of F_d will be a ball in $H(K_d)$; the second component will be the non-negative elements of a ball in $H(K_d)$. No assumption is made about the radii of these balls, other than that they must be independent of d.

Let

$$\kappa_2 = \int_0^1 \int_0^1 K(x, y)\, dx\, dy < \infty.$$

Since K is a reproducing kernel we know that $\kappa_2 \geq 0$. Our results depend on whether κ_2 is positive or zero, and whether we are dealing with the general case for finite-order weights of order ω or whether we are dealing with finite-order weights of order ω with a uniformly bounded sum, i.e., for which

$$\sup_{1 \leq d < \infty} \sum_{\substack{\mathfrak{u} \in \{1,\dots,d\} \\ |\mathfrak{u}| \leq \omega}} \gamma_{d,\mathfrak{u}} < \infty.$$

We may summarize our results as follows:

1. For absolute error criterion, we have

	General case		Bounded sum
	$\kappa_2 > 0$	$\kappa_2 = 0$	$\kappa_2 > 0$
Λ^{all}	$p_{\text{err}} \leq 2$, $p_{\text{dim}} \leq 2\omega$	$p_{\text{err}} \leq 2$, $p_{\text{dim}} \leq 3\omega$	$p_{\text{strong}} \leq 2$
Λ^{std}	$p_{\text{err}} \leq 4$, $p_{\text{dim}} \leq 4\omega$	$p_{\text{err}} \leq 2$, $p_{\text{dim}} \leq 6\omega$	$p_{\text{strong}} \leq 4$

2. For the normalized error criterion, we have

General case			Bounded sum
	$\kappa_2 > 0$	$\kappa_2 = 0$	$\kappa_2 > 0$
Λ^{all}	$p_{\text{err}} \le 2$, $p_{\text{dim}} \le \omega$	$p_{\text{err}} \le 2$, $p_{\text{dim}} \le 2\omega$	$p_{\text{strong}} \le 2$
Λ^{std}	$p_{\text{err}} \le 4$, $p_{\text{dim}} \le 2\omega$	$p_{\text{err}} \le 2$, $p_{\text{dim}} \le 4\omega$	$p_{\text{strong}} \le 4$

Hence, the heat equation is always tractable for finite-order weighted RKHSs, and it is strongly tractable if the sum of the weights is bounded.

It is worthwhile to compare the results for the heat equation with those we obtained in [16] for the Helmholtz equation:

1. The results for the heat equation under the absolute error criterion are the same as for the Helmholtz equation under both Dirichlet and Neumann boundary conditions.

2. The results for the heat equation under the normalized error criterion are the same as for the Helmholtz equation under Neumann boundary conditions.

Note that we studied both Dirichlet and Neumann boundary conditions in [16]. The main reason for introducing Neumann conditions in [16] was that we were unable to establish strong tractability for the Dirichlet problem under the normalized error criterion, and we wanted to exhibit a version of the problem for which the Neumann problem was strongly tractable. Since the Dirichlet problem for the heat equation is strongly tractable under the normalized error criterion if the weights have a bounded sum, we did not feel the need to analyze the Neumann problem for the heat equation. One advantage of this decision is that it greatly simplified the presentation.

2 The heat equation

We first establish a few notational conventions. For an ordered ring R, we let R^+ and R^{++} respectively denote the non-negative and positive elements of R. The open unit interval $(0, 1)$ is denoted by I. Since we are dealing with a time-dependent problem, we will let T denote a maximum time value. If X and Y are normed linear spaces, then $\text{Lin}[X, Y]$ denotes the space of bounded linear transformations of X into Y. We write $\text{Lin}[X]$ for $\text{Lin}[X, X]$, and X^* for $\text{Lin}[X, \mathbb{R}]$. For $\rho > 0$, we let $\mathscr{B}_\rho X$ denote the ball of radius ρ in X, centered at the origin, writing $\mathscr{B}X$ for the unit ball.

We use the standard notation for Sobolev inner products, seminorms, norms, and spaces, found in (e.g.) [6, 14]. Furthermore, for any normed linear space X, the spaces $C([0, T]; X)$, $L_2([0, T]; X)$, and $H^1([0, T]; X)$ are as defined in [7, pp. 381–382]. In particular, the norm of the space $C([0, T]; X)$ is given by

$$\|v\|_{C([0,T];X)} := \max_{0 \le t \le T} \|v(t)\|_X < \infty \qquad \forall \text{ continuous } v\colon [0, T] \to X.$$

For $d \in \mathbb{Z}^+$, we let Q_d denote the non-negative functions in $L_\infty(I^d)$. For $f \in L_2(I^d)$ and $q \in Q_d$, we wish to solve the parabolic partial differential equation

$$\dot{u}(t) = -(\mathscr{L}_q u)(t) \qquad (0 < t < T), \tag{1}$$

subject to the initial conditions

$$u(0) = f \tag{2}$$

and homogeneous boundary conditions

$$u = 0 \qquad \text{on } \partial I^d. \tag{3}$$

Here, the operator $\mathscr{L}_q \colon H^1_0(I^d) \to H^{-1}(I^d)$ is defined as

$$\mathscr{L}_q v = -\Delta v + qv \qquad \forall v \in H^1_0(I^d).$$

We shall refer to this problem as the *heat equation* in the rest of this paper.

Letting $\langle \cdot, \cdot \rangle$ denote the duality pairing of $H^1_0(I^d)$ with $H^{-1}(I^d)$, we have

$$\langle \mathscr{L}_q v, w \rangle = B_d(v, w; q) \qquad \forall\, v, w \in H^1_0(I^d).$$

Here, $B_d(\cdot, \cdot; q)$ is the bilinear form $H^1_0(I^d)$ given by

$$B_d(v, w; q) = \int_I^d [\nabla v \cdot \nabla w + qvw] \qquad \forall\, v, w \in H^1_0(I^d). \tag{4}$$

From [7, pp. 382–383], we have

Lemma 2.1. *For any $(f, q) \in L_2(I^d) \times Q_d$, there exists a unique solution*

$$u = S_d(f, q) \in L_2\bigl([0, T]; H^1_0(I^d)\bigr) \cap H^1\bigl([0, T]; H^{-1}(I^d)\bigr)$$

to the heat equation (1)–(3)*. Moreover, $u \in C\bigl([0, T]; L_2(I^d)\bigr)$.*

We next show that $S_d(f, q)$ depends continuously on f and q, this bound being sharp in its dependence on f.

Lemma 2.2. *Let $(f, q), (\tilde{f}, \tilde{q}) \in L_2(I^d) \times Q_d$. Then*

$$\begin{aligned}\|f - \tilde{f}\|_{L_2(I^d)} &\le \|S_d(f, q) - S_d(\tilde{f}, \tilde{q})\|_{C\left([0,T];L_2(I^d)\right)} \\ &\le \|f - \tilde{f}\|_{L_2(I^d)} + T\|q - \tilde{q}\|_{L_2(I^d)}\|f\|_{L_\infty(I^d)}.\end{aligned}$$

Proof. Let $u = S_d(f, q)$ and $\tilde{u} = S_d(\tilde{f}, \tilde{q})$. Since $u(0) = f$ and $\tilde{u}(0) = \tilde{f}$, we immediately obtain the first inequality. Hence, it only remains to prove the second inequality.

Without loss of generality, we shall assume that $u, \tilde{u} \in H^1_0(I^d)$. Choose $t \in (0, T)$, and let $e(t) = u(t) - \tilde{u}(t)$. Since $\mathscr{L}_q$ is self-adjoint in $L_2(I^d)$, we can check that

$$\langle \dot{e}(t), e(t) \rangle_{L_2(I^d)} = -B_d(e(t), e(t); \tilde{q}) + \langle (q - \tilde{q})u(t), e(t) \rangle_{L_2(I^d)}. \tag{5}$$

Since

$$\langle \dot{e}(t), e(t) \rangle_{L_2(I^d)} = \frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{L_2(I^d)} = \|e(t)\|_{L_2(I^d)}\frac{d}{dt}\|e(t)\|_{L_2(I^d)},$$

we may rewrite (5) as

$$\begin{aligned}\|e(t)\|_{L_2(I^d)}\frac{d}{dt}\|e(t)\|_{L_2(I^d)} &= -B_d(e(t), e(t); \tilde{q}) + \langle (q - \tilde{q})u(t), e(t) \rangle_{L_2(I^d)} \\ &\le \langle (q - \tilde{q})u(t), e(t) \rangle_{L_2(I^d)} \\ &\le \|(q - \tilde{q})u(t)\|_{L_2(I^d)}\|e(t)\|_{L_2(I^d)},\end{aligned}$$

where we have used the fact that $B_d(w, w; q) \ge 0$ for any $w \in H^1_0(I^d)$. Dividing the previous inequality by $\|e(t)\|_{L_2(I^d)}$, we find that

$$\frac{d}{dt}\|e(t)\|_{L_2(I^d)} \le \|(q - \tilde{q})u(t)\|_{L_2(I^d)}. \tag{6}$$

Recall (see, e.g., [2, Thm. 2.12]) that the strong maximum principle implies that

$$\|u(t)\|_{L_2(I^d)} \le \|f\|_{L_2(I^d)},$$

so that

$$\|(q-\tilde{q})u(t)\|_{L_2(I^d)} \le \|q-\tilde{q}\|_{L_2(I^d)}\|u(t)\|_{L_\infty(I^d)} \le \|q-\tilde{q}\|_{L_2(I^d)}\|f\|_{L_\infty(I^d)}.$$

Substituting this inequality into (6), we obtain

$$\frac{d}{dt}\|e(t)\|_{L_2(I^d)} \le \|q-\tilde{q}\|_{L_2(I^d)}\|f\|_{L_\infty(I^d)}.$$

Since we have the initial condition

$$\|e(0)\|_{L_2(I^d)} = \|f-\tilde{f}\|_{L_2(I^d)},$$

we find that

$$\|e(t)\|_{L_2(I^d)} \le \|f-\tilde{f}\|_{L_2(I^d)} + t\|q-\tilde{q}\|_{L_2(I^d)}\|f\|_{L_\infty(I^d)}.$$

Since $t \in (0,T)$ is arbitrary, this establishes the lemma. □

3 Information and algorithms

Let $F_d \subset L_2(I^d) \times Q_d$ be a set of *problem elements*. We want to approximate $S_d(f,q)$ for any $(f,q) \in F_d$, using finitely many values $f \mapsto \lambda(f)$ and $q \mapsto \lambda(q)$, where λ belongs to a class Λ of continuous linear functionals.

We shall restrict our attention to the following two choices for Λ:

1. *Continuous linear information*. This is the class Λ^{all} of all continuous linear functionals.

2. *Standard information*. This is the class Λ^{std} consisting of function evaluations. That is, $\lambda \in \Lambda^{\text{std}}$ if there exists $\mathbf{x}_\lambda \in \mathbb{R}^d$ such that $\lambda(g) = g(\mathbf{x}_\lambda)$ for any admissible function g.

Recall that $d \in \mathbb{Z}^{++}$ is the number of variables on which our input functions f and q and our solution u depend. Given $n \in \mathbb{Z}^{++}$, let $A_{d,n}$ be an algorithm for approximating S_d, using at most n information evaluations from a class Λ. The worst case *error* of $A_{d,n}$ is defined to be

$$e(A_{d,n}, S_d, F_d, \Lambda) = \sup_{[f,q]\in F_d} \|S_d(f,q) - A_{d,n}(f,q)\|_{C\left([0,T];L_2(I^d)\right)}.$$

The nth *minimal error* is defined to be

$$e(n, S_d, F_d, \Lambda) = \inf_{A_{d,n}} e(A_{d,n}, S_d, F_d, \Lambda),$$

the infimum being over all algorithms using at most n information evaluations from Λ.

In particular, note that $e(0, S_d, F_d, \Lambda)$ is the *initial error*, which is obtained without using *any* information evaluations whatsoever. Since this initial error involves no information evaluations, it is independent of Λ, and hence we shall simply denote it as $e(0, S_d, F_d)$.

Let $\varepsilon \in (0, 1)$. We wish to measure the minimal number of information evaluations needed to compute an ε-approximation. Here, we say that an algorithm $A_{d,n}$ provides an *ε-approximation* to S_d if

$$e(A_{d,n}, S_d, F_d, \Lambda) \le \varepsilon \cdot \mathrm{ErrCrit}(S_d, F_d),$$

with ErrCrit being an *error criterion*. In this paper, we will use the error criteria

$$\mathrm{ErrCrit}(S_d, F_d) = \begin{cases} 1 & \text{for absolute error,} \\ e(0, S_d, F_d) & \text{for normalized error.} \end{cases}$$

Hence:

1. An algorithm provides an ε-approximation in the *absolute* sense simply means that the error of the algorithm is at most ε.

2. An algorithm provides an ε-approximation in the *normalized* sense simply means that the error of the algorithm is *reduced* by at least a factor of ε.

In either case, let

$$\mathrm{card}(\varepsilon, S_d, F_d, \Lambda) = \min\left\{ n \in \mathbb{Z}^+ : e(\varepsilon, S_d, F_d, \Lambda) \le \varepsilon \cdot \mathrm{ErrCrit}(S_d, F_d) \right\}$$

denote the minimal number of information evaluations needed to compute an ε-approximation to S_d. Of course, the ε-cardinalities for the absolute and normalized criteria are related by the equation

$$\mathrm{card}^{\mathrm{nor}}(\varepsilon, S_d, F_d, \Lambda) = \mathrm{card}^{\mathrm{abs}}(\varepsilon \cdot e(0, S_d, F_d), S_d, F_d, \Lambda). \tag{7}$$

As mentioned in the Introduction, we often want to solve heat equations of high dimension. The heat equation is said to be *tractable* with respect to the class Λ of information functionals if there exist non-negative numbers C, p_{err}, and p_{dim} such that

$$\mathrm{card}(\varepsilon, S_d, F_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\mathrm{err}}} d^{p_{\mathrm{dim}}} \qquad \forall\, \varepsilon \in (0, 1), d \in \mathbb{Z}^{++}. \tag{8}$$

(If (8) does not hold, then the problem is said to be *intractable*.) Any numbers $p_{\mathrm{err}} = p_{\mathrm{err}}(\Lambda)$ and $p_{\mathrm{dim}} = p_{\mathrm{dim}}(\Lambda)$ such that (8) holds are called *ε- and d-exponents of tractability*. These exponents need not be uniquely defined. If $p_{\mathrm{dim}} = 0$ in (8), then the heat equation is said to be *strongly tractable* with respect to Λ, and we define

$$p_{\mathrm{strong}}(\Lambda) = \\ \inf\left\{ p_{\mathrm{err}} \ge 0 : \ \exists\, C \ge 0 \text{ such that } \mathrm{card}(\varepsilon, S_d, F_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\mathrm{err}}} \forall\, \varepsilon \in (0, 1), d \in \mathbb{Z}^{++} \right\}$$

to be the *exponent of strong tractability*.

4 Intractability for classical Sobolev spaces

Recall that our set F_d of problem elements is a subset of $L_2(I^d) \times Q_d$, where Q_d denotes the non-negative elements of $L_\infty(I^d)$. We briefly discuss tractability when the first component of F_d is a ball of fixed radius in a standard Sobolev space. There is no essential loss of generality in assuming that this ball has unit radius.

We first consider arbitrary continuous linear information.

Theorem 4.1. *Let $\Lambda = \Lambda^{\text{all}}$. Regardless of whether the absolute or normalized error criterion is used, the heat equation is intractable if the first component of F_d is $\mathscr{B}H^r(I^d)$.*

Proof. First, suppose that we are using the absolute error criterion. From the lower bound in Lemma 2.2, we see that

$$e(n, S_d, F_d, \Lambda^{\text{all}}) \geq e(n, \text{App}_d, \mathscr{B}H^r(I^d), \Lambda^{\text{all}}),$$

where $\text{App}_d\colon H^r(I^d) \to L_2(I^d)$ is the approximation problem given by

$$\text{App}_d\, f = f \qquad \forall f \in H^r(I^d).$$

It is well-known (see, e.g., [4]) that there exists $C_d > 0$ such that

$$e(n, \text{App}_d, \mathscr{B}H^r(I^d), \Lambda^{\text{all}}) \geq C_d n^{-r/d}.$$

Combining these results, we see that

$$\text{card}^{\text{abs}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}}) \geq \left(\frac{C_d}{\varepsilon}\right)^{d/r},$$

and hence our problem is intractable in the absolute error criterion.

We now turn to the normalized error criterion. Fix $(f, q) \in F_d$, and let $u = S_d(f, q)$. For any $t \in [0, T]$, we have the series representation

$$u(t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle f, z_j \rangle_{L_2(I^d)} z_j,$$

where $z_1, z_2, \dots \in H_0^1(I^d)$ are the $L_2(I)$-orthonormal eigenvectors of $\mathscr{L}_q$ corresponding to the positive eigenvalues $\lambda_1 \leq \lambda_2 \dots$, from which we see that

$$\|u(t)\|_{L_2(I^d)} \leq \|f\|_{L_2(I^d)} \leq \|f\|_{H^r(I^d)}.$$

Since $S_d(\cdot, q) \in \text{Lin}[H^r(I^d), L_2(I^d)]$ for any $q \in Q_d$, we may use the results of [9, §4.5], along with the previous inequality, to find that

$$e(0, S_d, F_d, \Lambda^{\text{all}}) = \max_{0 \leq t \leq T} \sup_{(f,q) \in F_d} \|S_d(f, q)(t)\|_{L_2(I^d)} \leq 1.$$

Hence

$$\frac{e(n, S_d, F_d, \Lambda^{\text{all}})}{e(0, S_d, F_d, \Lambda^{\text{all}})} \geq C_d n^{-r/d},$$

and so we have

$$\text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}}) \geq \left(\frac{C_d}{\varepsilon}\right)^{d/r}.$$

Thus our problem is intractable in the normalized error criterion. $\square$

Remark. Note that we are approximating the solution of the heat equation over the time interval $[0, T]$. The solution $S_d(f, q)$ at time $t = 0$ is simply f, and so L_2-approximation problem is a special case of our problem. Since the latter problem is intractable over the unit ball of $H^r(I^d)$, our heat equation is intractable when f belongs to $\mathscr{B}H^r(I^d)$.

One might well ask what would happen if we were only trying to approximate the solution at a fixed positive time value t. It turns out that our problem is still intractable. Indeed, let $S_{d,t} = S_d(\cdot, \cdot)(t)$ be the solution operator at time t. Define

$$\tilde{F}_d = \mathscr{B}\dot{H}^r(I^d) \times \{0\},$$

where $\dot{H}^r(I^d)$ is the span of the $\mathscr{L}_q$-eigenvectors $\{z_j\}_{j=1}^{\infty}$ under the norm $\|\cdot\|_{\dot{H}^r(I^d)} = \|L_0^{r/2} \cdot\|_{L_2(I^d)}$. Then

$$e(n, S_{d,t}, F_d, \Lambda^{\text{all}}) \succcurlyeq e(n, S_{d,t}, \tilde{F}_d, \Lambda^{\text{all}}).$$

In this case, it is possible to use the techniques of [13] to see that

$$e(n, S_{d,t}, \tilde{F}_d, \Lambda^{\text{all}}) = \lambda_{n+1}^{-r/2} e^{-\lambda_{n+1} t} \sim (n+1)^{-r/d} e^{-c_d (n+1)^{2/d} t}$$

for a positive constant c_d. It is fairly easy to see that

$$\text{card}^{\text{abs}}(\varepsilon, S_{d,t}, \tilde{F}_d, \Lambda^{\text{all}}) \sim \left(\frac{1}{t c_d} \ln \frac{1}{\varepsilon} \right)^{d/2}.$$

Hence we find that

$$\text{card}(\varepsilon, S_{d,t}, F_d, \Lambda^{\text{all}}) \succcurlyeq \left(\frac{1}{t c_d} \ln \frac{1}{\varepsilon} \right)^{d/2}$$

for either the absolute or normalized error criterion. Hence approximating the heat equation at a fixed time $t > 0$ is intractable if the first component of F_d is a standard Sobolev space of fixed smoothness. $\square$

What can we say regarding standard information? Recall that the first component of our problem element class F_d is the unit ball of $H^r(I^d)$. The Sobolev embedding theorem tells us that evaluating f at a point in I^d is not well-defined for all $f \in H^r(I^d)$ unless $r > d/2$. In other words, standard information is ill-defined unless $r > d/2$. Since we need $r > d/2$ to even talk about nontrivial algorithms using standard information, we see that it is impossible to compute an ε-approximation for fixed r if $d \geq 2r$. This is a most severe form of intractability.

5 Weighted reproducing kernel Hilbert spaces

Since the heat equation is intractable for standard Sobolev spaces, we need to choose a different space of problem elements if we want our problem to be tractable. More precisely, we shall assume that our problem elements come from a weighted reproducing kernel Hilbert space (RKHS) $H(K_d)$ of functions defined over I^d. In this section, we briefly recall the definition of a weighted RKHS. This summary is essentially the same as that contained in [16, §2]; we include it for the convenience of the reader.

Let K be a reproducing kernel defined on $I \times I$. We will require that

$$\kappa_0 := \operatorname*{ess\,sup}_{x \in I} K(x, x) < \infty, \tag{9}$$

from which it follows that

$$0 \leq \kappa_2 \leq \kappa_1 \leq \kappa_0,$$

where

$$\kappa_1 = \int_0^1 K(x,x)\,dx \tag{10}$$

and

$$\kappa_2 = \int_0^1 \int_0^1 K(x,y)\,dy\,dx. \tag{11}$$

We now give some examples of commonly-occurring kernels.

Example. The *min kernel* is defined as

$$K_{\min}(x,y) := \min\{x,y\} \qquad \forall\, x,y \in [0,1]. \tag{12}$$

The space $H(K_{\min})$ consists of absolutely continuous functions vanishing at zero and whose first derivatives belong to $L_2(I)$, with the inner product

$$\langle f,g\rangle_{H(K_{\min})} = \int_I f'(x)g'(x)\,dx.$$

It is easy to check that we have

$$\begin{aligned}\kappa_0 &= 1\\ \kappa_1 &= \tfrac{1}{2}\\ \kappa_2 &= \tfrac{1}{3}\end{aligned}$$

for the min kernel. □

Example. The *Korobov kernel* is defined as

$$K_{\mathrm{Kor}}(x,y) := B_2(|x-y|) \qquad \forall\, x,y \in [0,1], \tag{13}$$

where $B_2(t) = t^2 - t + \frac{1}{6}$ is the Bernoulli polynomial of degree 2. The space $H(K_{\mathrm{Kor}})$ consists of absolutely continuous functions whose average value is zero and whose first derivatives belong to $L_2(I)$, with the inner product

$$\langle f,g\rangle_{H(K_{\mathrm{Kor}})} = \int_I f'(x)g'(x)\,dx.$$

It is easy to check that we have

$$\begin{aligned}\kappa_0 &= \tfrac{1}{6}\\ \kappa_1 &= \tfrac{1}{6}\\ \kappa_2 &= 0\end{aligned}$$

for the Korobov kernel. □

Remark. Note that the spaces $H(K_{\min})$ and $H(K_{\mathrm{Kor}})$ are both spaces of $H^1(I)$-functions with the same inner product. The only difference between them is that $H(K_{\min})$-functions vanish at the endpoints of I, whereas $H(K_{\mathrm{Kor}})$-functions have zero average value. The fact that $\kappa_2 > 0$ for $H(K_{\min})$, whereas $\kappa_2 = 0$ for $H(K_{\mathrm{Kor}})$, will greatly affect the tractability results for the corresponding spaces $H(K_{d,\min})$ and $H(K_{d,\mathrm{Kor}})$. See [8] for further properties of these (and similar) spaces. □

We now move on to the d-variate case. Let

$$\gamma = \{ \gamma_{d,\mathfrak{u}} : \mathfrak{u} \in \mathscr{P}_d, d \in \mathbb{Z}^{++} \}$$

be a set of non-negative weights, with

$$\gamma_{\max} := \max_{d \in \mathbb{Z}^{++}} \max_{\mathfrak{u} \in \mathscr{P}_d} \gamma_{d,\mathfrak{u}} < \infty.$$

We shall assume that γ is a set of *finite-order* weights (see, e.g., [1]), which means that there exists $\omega \in \mathbb{Z}^{++}$ such that

$$\gamma_{d,\mathfrak{u}} = 0 \qquad \forall\, \mathfrak{u} \in \mathscr{P}_d \text{ and } |\mathfrak{u}| > \omega,\ d \in \mathbb{Z}^{++}. \tag{14}$$

The *order* of a set γ of finite-order weights is the smallest $\omega \in \mathbb{Z}^{++}$ such that (14) holds.

The space $H(K_d)$ is the reproducing kernel Hilbert space (RKHS) whose reproducing kernel is

$$K_d = \sum_{\mathfrak{u} \in \mathscr{P}_d} \gamma_{d,\mathfrak{u}} K_{d,\mathfrak{u}},$$

where $\mathscr{P}_d$ is the power set of $\{1, \ldots, d\}$ and

$$K_{d,\mathfrak{u}}(\mathbf{x}, \mathbf{y}) = \prod_{j \in \mathfrak{u}} K(x_j, y_j) \qquad \forall\, \mathbf{x} = [x_1, \ldots, x_d], \mathbf{y} = [y_1, \ldots, y_d] \in \bar{I}^d, \mathfrak{u} \in \mathscr{P}_d.$$

Equivalently, $H(K_d)$ consists of those functions $f : I^d \to \mathbb{R}$ that can be uniquely decomposed as

$$f(\mathbf{x}) = \sum_{\mathfrak{u} \in \mathscr{P}_d,\, |\mathfrak{u}| \le \omega} f_{\mathfrak{u}}(\mathbf{x}),$$

where $f_{\mathfrak{u}}(\mathbf{x}) = f(\mathbf{x}_{\mathfrak{u}})$ depends only on x_j for $j \in \mathfrak{u}$, and $f_{\mathfrak{u}} \in H(K_{d,\mathfrak{u}})$. Furthermore

$$\|f\|^2_{H(K_d)} = \sum_{\mathfrak{u} \in \mathscr{P}_d,\, |\mathfrak{u}| \le \omega} \gamma_{d,\mathfrak{u}}^{-1} \|f_{\mathfrak{u}}\|^2_{H(K_{d,\mathfrak{u}})},$$

where

$$\|f_{\mathfrak{u}}\|^2_{H(K_{d,\mathfrak{u}})} = \int_{I^{|\mathfrak{u}|}} \left(\frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}_{\mathfrak{u}}) \right)^2 d\mathbf{x}_{\mathfrak{u}}.$$

Here, by convention, we have $0/0 = 0$. That is, if $\gamma_{d,\mathfrak{u}} = 0$, then the corresponding component $f_{\mathfrak{u}} = 0$.

Observe that the constant function $f(\mathbf{x}) = c$ for all $\mathbf{x} \in I^d$ belongs to $H(K_d)$ iff $\gamma_{d,\emptyset} > 0$, in which case we have $\|f\|_{H(K_d)} = |c|/\gamma_{d,\emptyset}^{1/2}$.

In what follows, it will be useful to let

$$\sigma_d(\theta) = \left(\sum_{\mathfrak{u} \in \mathscr{P}_d} \gamma_{d,\mathfrak{u}} \theta^{|\mathfrak{u}|} \right)^{1/2} \qquad \forall\, \theta \in \mathbb{R}^+. \tag{15}$$

For $g \in H(K_d)$, we know that

$$\|g\|_{L_2(I^d)} \le \sigma_d(\kappa_1) \|g\|_{H(K_d)} \tag{16}$$

and that

$$\|g\|_{L_\infty(I^d)} \le \sigma_d(\kappa_0) \|g\|_{H(K_d)} \qquad \forall\, g \in H(K_d), \tag{17}$$

see [11] and [16, Lemma 3.1]. Hence, $H(K_d)$ is embedded in $L_2(I^d)$ and $L_\infty(I^d)$ for arbitrary weights γ, and we know values for the embedding constants. For finite-order weights of order ω, we can estimate $\sigma_d(\theta)$ by

$$\sigma_d(\theta) \le \sqrt{2 \max\{\theta^\omega, 1\}\, \gamma_{\max}}\, d^{\,\omega/2} \tag{18}$$

see [15, Lemma 6].

6 Tractability for weighted RKHS

In the remainder of this paper, we shall assume that our problem elements belong to a weighted RKHS. More precisely, we shall assume that

$$F_d = H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2})$$

for fixed positive ρ_1 and ρ_2, where (for the sake of convenience) we write $H_{d,\rho} = \mathscr{B}_\rho K(H_d)$ for any $\rho > 0$. Hence we are trying to approximate $S_d(f, q)$ for $f \in H_{d,\rho_1}$ and $q \in Q \cap H_{d,\rho_2}$.

6.1 Some preliminary results

We will establish tractability of the heat equation by using the results of [15], which gives a mechanism for establishing the (strong) tractability of quasilinear problems defined over a weighted RKHS. Here (as in [15]) we say that our problem $\{S_d\}_{d=1}^{\infty}$ is *quasilinear* if there exists a function $\phi\colon H(K_d) \to Q_d$, as well as a non-negative number C_d, such that

$$\|S_d(f,q) - S_d(\tilde{f}, \phi(\tilde{q}))\|_{G_d} \le C_d \left[\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \tilde{q}\|_{L_2(I^d)} \right] \qquad \forall\, [f,q] \in H_{d,\rho_1} \times Q_d,\ [\tilde{f}, \tilde{q}] \in H(K_d) \times H(K_d). \tag{19}$$

Our first preliminary result establishes that the heat equation is quasilinear. Let us define $\phi\colon H(K_d) \to Q_d$ as

$$\phi(v)(\mathbf{x}) = v_+(\mathbf{x}) := \max\{v(\mathbf{x}), 0\} \qquad \forall\, \mathbf{x} \in I^d,\ v \in H(K_d). \tag{20}$$

Lemma 6.1. *Let*

$$C_d = \max\{1, \rho_1 T \sigma_d(\kappa_0)\},$$

where κ_0 is given by (9). *Then the heat equation problem $\{S_d\}_{d=1}^{\infty}$ is quasilinear for ϕ given by* (20).

Proof. Let $(f, q), (\tilde{f}, \tilde{q}) \in F_d$. As in [16, Lemma 3.4], we find that

$$\|q - \phi(\tilde{q})\|_{L_2(I^d)} \le \|q - \tilde{q}\|_{L_2(I^d)}.$$

Using (17) and Lemma 2.2, we see that (19) holds, as required. □

Suppose that there exists $\alpha \ge 0$ such that

$$N_\alpha := \sup_{d \in \mathbb{Z}^{++}} \frac{C_d \| \operatorname{App}_d \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]}}{d^\alpha \operatorname{ErrCrit}(S_d)} < \infty. \tag{21}$$

where $\operatorname{App}_d\colon H(K_d) \to L_2(I^d)$ is now the embedding

$$\operatorname{App}_d f = f \qquad \forall\, f \in H(K_d).$$

Then [15, Theorem 3] tells us that the problem is tractable if $\alpha > 0$ and strongly tractable if $\alpha = 0$. More precisely, [15, Theorem 3] provides algorithms for computing an ε-approximation of S_d and having an error bound $C(1/\varepsilon)^{p_{\text{err}}} d^{p_{\text{dim}}}$ (for tractability) or $C(1/\varepsilon)^{p_{\text{strong}}}$ (for strong tractability), along with explicit expressions for C, p_{err}, p_{dim}, and p_{strong}.

One of the most important parts of the analysis will be to determine the minimal α such that (21) holds. To do this, we will need to estimate the norm of App_d. Note that (16) implies that the embedding App_d is well-defined, with

$$\| \mathrm{App}_d \|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} \le \sigma_d(\kappa_1). \tag{22}$$

More precise results for $\| \mathrm{App}_d \|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}$ are given in [11]. For the case $\kappa_2 = 0$, these results involve the operator $W = (\mathrm{App})^*(\mathrm{App}) \in \mathrm{Lin}[H(K)]$, where App is the embedding operator $\mathrm{App} \in \mathrm{Lin}[H(K), L_2(I)]$. Note that

$$Wf = \int_0^1 K(x,\cdot) f(x)\, dx \qquad \forall f \in H(K) \tag{23}$$

and that

$$\|W\|_{\mathrm{Lin}[H(K)]} = \| \mathrm{App} \|^2_{\mathrm{Lin}[H(K),L_2(I)]} \le \kappa_1. \tag{24}$$

We then have

Lemma 6.2. *Let κ_1, κ_2 and σ_d be defined by* (10), (11) *and* (15).

1. *There exists $c_d \in [\kappa_2, \kappa_1]$ such that*

$$\| \mathrm{App}_d \|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} = \sigma_d(c_d).$$

2. *If $\kappa_2 = 0$, then*

$$\| \mathrm{App}_d \|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} = \max_{\mathfrak{u} \in \mathscr{P}_d} \left[\gamma_{d,\mathfrak{u}} \|W\|^{\mathfrak{u}}_{\mathrm{Lin}[H(K)]} \right]^{1/2}.$$

Remark. Recall that $\kappa_1 = \frac{1}{6}$ and $\kappa_2 = 0$ for the Korobov kernel. With an eye towards future results, we note that

$$\|W\|_{\mathrm{Lin}[H(K_{\mathrm{Kor}})]} = \sup \left\{ \frac{\int_I f(x)^2\, dx}{\int_I \left(f'(x)\right)^2 dx} : f \in H^1(I) \text{ such that } \int_I f(x)\, dx = 0 \right\}.$$

Choosing f to be the function

$$f(x) = x - \tfrac{1}{2} \qquad \forall x \in I,$$

and using (24), we find that

$$\tfrac{1}{12} \le \|W\|_{\mathrm{Lin}[H(K_{\mathrm{Kor}})]} \le \tfrac{1}{6}. \tag{25}$$

The following result (also from [11]) gives two useful algorithms for the approximation problem App_d, which will be used as building blocks of algorithms for the heat equation:

Lemma 6.3. *Let $d \in \mathbb{Z}^{++}$ and $n \in \mathbb{Z}^{+}$.*

1. *Let*

$$A^*_{d,n}(f) = \sum_{j=1}^{n} \langle f, e_{d,j} \rangle_{H(K_d)} e_{d,j} \qquad \forall f \in H(K_d).$$

Then

$$\| \mathrm{App}_d - A^*_{d,n} \|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} \le \frac{\sigma_d(\kappa_1)}{\sqrt{n+1}}.$$

2. *There exist points* $\mathbf{t}_1, \ldots, \mathbf{t}_n$ *and elements* $a_1, \ldots, a_n \in H(K_d)$ *such that*

$$A_{d,n}(f) = \sum_{j=1}^{n} f(\mathbf{t}_j) a_j \qquad \forall f \in H(K_d),$$

we have

$$\| \operatorname{App}_d - A_{d,n} \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]} \le \frac{\sigma_d(\kappa_1)\sqrt{2}}{n^{1/4}}.$$

We stress that the results in part 2 of Lemma 6.3 are non-constructive for the class Λ^{std}, i.e., we only know that there exist points $\mathbf{t}_1, \ldots \mathbf{t}_n$ such that the algorithm $A_{d,n}$ has the given error bound. Weaker constructive error bounds may be found in [12].

Using these algorithms $A^*_{d,n}$ and $A_{d,n}$, we define

$$U^*_{d,n}(f, q) = S_d\left(A^*_{d,\lfloor n/2 \rfloor} f, \phi(A^*_{d,\lfloor n/2 \rfloor} q)\right) \qquad \forall [f, q] \in H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2})$$

and

$$U_{d,n}(f, q) = S_d\left(A_{d,\lfloor n/2 \rfloor} f, \phi(A_{d,\lfloor n/2 \rfloor} q)\right) \qquad \forall [f, q] \in H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2}).$$

Clearly, $U^*_{d,n}$ and $U_{d,n}$ are algorithms for the heat equation using continuous linear information and standard information, respectively.

6.2 Results for the absolute error criterion

Since $\mathrm{ErrCrit}(S_d) = 1$ for the absolute error criterion, finding α for which (21) is satisfied means that we need to determine α such that $C_d \| \operatorname{App}_d \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]}$ is of order d^α.

Theorem 6.1. *The heat equation for* $H(K_d)$ *with finite-order weights of order* ω *is tractable for the absolute error. More precisely, for* N_ω *defined by* (21), *we have*

$$N_\omega \le \max\left\{1, \rho_1 T \sqrt{2 \max\{1, \kappa_0^{\omega}\} \gamma_{\max}}\right\} \sqrt{2 \max\{1, \kappa_1^{\omega}\} \gamma_{\max}}\,, \tag{26}$$

and the following bounds hold:

1. *Suppose that* $\kappa_2 > 0$.

   (a) *For the class* Λ^{all}, *we have*

   $$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{all}}) \le 2(\rho_1 + \rho_2)^2 N_\omega^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} \left(\frac{1}{\varepsilon}\right)^2 d^{2\omega}.$$

   Moreover, the algorithm $U^*_{d,n}$, *with* $n = \mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F, d\Lambda^{\mathrm{all}})$, *gives an* ε*-approximation. Hence*

   $$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{all}}) \le 2 \qquad \textit{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{all}}) \le 2\omega.$$

   (b) *For the class* Λ^{std}, *we have*

   $$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1 + \rho_2)^4 N_\omega^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{4\omega} \right\rceil + 1.$$

   Moreover, the algorithm $U_{d,n}$, *with* $n = \mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d \Lambda^{\mathrm{std}})$, *gives an* ε*-approximation. Hence*

   $$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{std}}) \le 4 \qquad \textit{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{std}}) \le 4\omega.$$

2. *Suppose that $\kappa_2 = 0$, and let*

$$\Gamma = \frac{\max\{1, \kappa_1\}}{\min\{1, \|W\|_{\mathrm{Lin}[H(K)]}\}}. \tag{27}$$

Then we have the following results:

(a) *For the class Λ^{all}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{all}}) \le 4(\rho_1 + \rho_2)^2 N_\omega^2 \Gamma^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{3\omega}.$$

Moreover, the algorithm $U^*_{d,n}$, with $n = \mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{all}})$, gives an ε-approximation. Hence

$$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{all}}) \le 2 \qquad \textit{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{all}}) \le 3\omega.$$

(b) *For the class Λ^{std}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{std}}) \le \left\lceil 32(\rho_1 + \rho_2)^4 N_\omega^4 \Gamma^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{6\omega} \right\rceil + 1.$$

Moreover, the algorithm $U_{d,n}$, with $n = \mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{std}})$, gives an ε-approximation. Hence

$$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{std}}) \le 4 \qquad \textit{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{std}}) \le 6\omega.$$

Proof. Using (18), (22), and Lemma 6.1, we find that

$$C_d \|\,\mathrm{App}_d\,\|_{\mathrm{Lin}[H(K_d), L_2(I^d)]} \le \max\left\{1, \rho_1 T \sqrt{2\max\{1, \kappa_0^{(\omega)}\}\gamma_{\max}}\right\} \sqrt{2\max\{1, \kappa_1^{(\omega)}\}\gamma_{\max}} \cdot d^\omega.$$

Hence setting $\alpha = \omega$ in (21), we obtain (26). The remaining results of this theorem now follow from [15, Theorem 7], with $\alpha = \omega$. $\square$

Example. Suppose that K is the min kernel $K_{\min}$. Since $\kappa_0 = 1$ and $\kappa_1 = \frac{1}{2}$, we have

$$N_\omega \le \max\left\{1, \rho_1 T \sqrt{2\gamma_{\max}}\right\} \sqrt{2\gamma_{\max}}$$

from (26). Furthermore, since $\kappa_2 = \frac{1}{3} \ne 0$, we see that case 1 holds in Theorem 6.1. Hence we find that the heat equation is now tractable under the absolute error criterion, with

$$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{all}}) \le 2 \qquad \text{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{all}}) \le 2\omega,$$

for continuous linear information and

$$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{std}}) \le 4 \qquad \text{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{std}}) \le 4\omega$$

for standard information. $\square$

Example. Now suppose that K is the min kernel K_{Kor}. Since $\kappa_0 = \kappa_1 = \frac{1}{6}$, we again have

$$N_\omega \le \max\left\{1, \rho_1 T\sqrt{2\gamma_{\max}}\right\}\sqrt{2\gamma_{\max}}$$

from (26). Furthermore, since $\kappa_2 = 0$, we see that case 2 holds in Theorem 6.1, with $\Gamma \le 12$ by (25). Hence we find that the heat equation is now tractable under the absolute error criterion, with

$$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{all}}) \le 2 \qquad \text{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{all}}) \le 3\omega,$$

for continuous linear information and

$$p^{\mathrm{abs}}_{\mathrm{err}}(\Lambda^{\mathrm{std}}) \le 4 \qquad \text{and} \qquad p^{\mathrm{abs}}_{\mathrm{dim}}(\Lambda^{\mathrm{std}}) \le 6\omega$$

for standard information. □

Theorem 6.1 tells us that the heat equation is tractable under the absolute error criterion for any finite-order weighted RKHS, no matter what set of weights is used. The reason we are unable to establish strong tractability in this case is that the Lipschitz constant C_d and $\| \mathrm{App}_d \|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}$ are expressed in terms of $\sigma_d(\kappa_0)$ and $\sigma(\kappa_1)$, whose product is bounded by a polynomial of degree ω in d. Hence we can only guarantee that N_ω is finite. It is proved in [15, Theorem 7] that strong tractability holds for a quasilinear problem if $\kappa_2 > 0$ and if N_0 is finite. We can guarantee that N_0 is finite if we follow the approach taken in [15, Theorem 8].

Theorem 6.2. *Suppose that $\kappa_2 > 0$ and*

$$\rho_3 := \sup_{d\in\mathbb{Z}^{++}} \sum_{u\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} < \infty. \tag{28}$$

The heat equation for $H(K_d)$ with finite-order weights of order ω satisfying (28) *is strongly tractable for the absolute error. More precisely, for N_0 defined by* (21), *we have*

$$N_0 \le \rho_3^{1/2} \max\{1, \kappa_1^{\omega/2}\} \max\left\{1, \rho_1\rho_3^{1/2} T \max\{1, \kappa_0^{\omega/2}\}\right\}, \tag{29}$$

and the following bounds hold:

1. *For the class Λ^{all}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{all}}) \le 2(\rho_1+\rho_2)^2 N_0^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} \left(\frac{1}{\varepsilon}\right)^2,$$

Moreover, the algorithm $U^*_{d,n}$, with $n = \mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{all}})$, gives an ε-approximation. Hence

$$p^{\mathrm{abs}}_{\mathrm{strong}}(\Lambda^{\mathrm{all}}) \le 2.$$

2. *For the class Λ^{std}, we have*

$$\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1+\rho_2)^4 N_0^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 \right\rceil + 1.$$

Moreover, the algorithm $U_{d,n}$, with $n = \mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, F_d, \Lambda^{\mathrm{std}})$, gives an ε-approximation. Hence

$$p^{\mathrm{abs}}_{\mathrm{strong}}(\Lambda^{\mathrm{std}}) \le 4.$$

Proof. Using (28), it follows that

$$\sigma_d(\theta) \;\le\; \rho_3^{1/2} \max\{1, \theta^{\omega/2}\} \qquad \forall\, \theta \in \mathbb{R}^+. \tag{30}$$

From (21), (22), and (30), we have

$$N_0 \le C^* \rho_3^{1/2} \max\{1, \kappa_1^{\omega/2}\},$$

where

$$C^* = \sup_{d\in\mathbb{Z}^{++}} C_d = \max\left\{1, \rho_1 T \sup_{d\in\mathbb{Z}^{++}} \sigma_d(\kappa_0)\right\} \le \max\left\{1, \rho_1 T \cdot \rho_3^{1/2} \max\{1, \kappa_0^{\omega/2}\}\right\}$$

by Lemma 6.1 and (30). Combining these results, we obtain (29). The desired result now follows from [15, Theorem 8]. □

Example. Suppose once again that $K = K_{\min}$. Assume that (28) holds. Then the conditions of Theorem 6.2 are satisfied with

$$N_0 \le \rho_3^{1/2} \max\{1, \rho_1 \rho_3^{1/2} T\} \qquad \text{and} \qquad \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} = \left(\frac{3}{2}\right)^{\omega}.$$

Hence, the heat is now strongly tractable under the absolute error criterion, with

$$p^{\text{abs}}_{\text{strong}}(\Lambda^{\text{all}}) \le 2 \qquad \text{and} \qquad p^{\text{abs}}_{\text{strong}}(\Lambda^{\text{all}}) \le 4. \quad \square$$

6.3 Results for the normalized error criterion

We have $\operatorname{ErrCrit}(S_d) = e(0, S_d)$ for the normalized error criterion. Moreover, since $S_d(f, q)(0) = f$ for any $(f, q) \in F_d$, it is clear that

$$\begin{aligned} e(0, S_d) &= \max_{t\in[0,T]} \sup_{(f,q)\in F_d} \|S_d(f, q)(t)\|_{|L_2(I^d)} \ge \sup_{(f,q)\in F_d} \|S_d(f, q)(0)\|_{|L_2(I^d)} \\ &= \sup_{f\in H_{d,\rho_1}} \|f\|_{|L_2(I^d)} = \|\operatorname{App}_d\|_{\operatorname{Lin}[H(K_d), L_2(I^d)]}. \end{aligned}$$

Hence, we can find α for which (21) is satisfied if we can determine α such that C_d is of order d^α.

Theorem 6.3. *The heat equation for $H(K_d)$ with finite-order weights of order $\omega/2$ is tractable for the normalized error. More precisely, for $N_{\omega/2}$ defined by* (21)*, we have*

$$N_{\omega/2} \le \max\left\{1, \rho_1 T \sqrt{2 \max\{1, \kappa_0^{\omega}\}\gamma_{\max}}\right\}, \tag{31}$$

and the following bounds hold:

1. *Suppose that $\kappa_2 > 0$.*

   (a) *For the class Λ^{all}, we have*

   $$\operatorname{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}}) \le 2(\rho_1 + \rho_2)^2 N^2_{\omega/2} \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} \left(\frac{1}{\varepsilon}\right)^2 d^{\omega}.$$

   Moreover, the algorithm $U^*_{d,n}$, with $n = \operatorname{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}})$, gives an ε-approximation. Hence

   $$p^{\text{nor}}_{\text{err}}(\Lambda^{\text{all}}) \le 2 \qquad \textit{and} \qquad p^{\text{nor}}_{\text{dim}}(\Lambda^{\text{all}}) \le \omega.$$

(b) *For the class* Λ^{std}*, we have*

$$\text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{std}}) \le \left\lceil 8(\rho_1+\rho_2)^4 N_{\omega/2}^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{2\omega} \right\rceil + 1.$$

Moreover, the algorithm $U_{d,n}$*, with* $n = \text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{std}})$*, gives an* ε*-approximation. Hence*

$$p_{\text{err}}^{\text{nor}}(\Lambda^{\text{std}}) \le 4 \qquad \textit{and} \qquad p_{\text{dim}}^{\text{nor}}(\Lambda^{\text{std}}) \le 2\omega.$$

2. *Suppose that* $\kappa_2 = 0$*, and let* Γ *be defined by* (27)*. Then we have the following results:*

(a) *For the class* Λ^{all}*, we have*

$$\text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}}) \le 4(\rho_1+\rho_2)^2 N_{\omega/2}^2 \Gamma^{\omega} \left(\frac{1}{\varepsilon}\right)^2 d^{2\omega}.$$

Moreover, the algorithm $U_{d,n}^*$*, with* $n = \text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}})$*, gives an* ε*-approximation. Hence*

$$p_{\text{err}}^{\text{nor}}(\Lambda^{\text{all}}) \le 2 \qquad \textit{and} \qquad p_{\text{dim}}^{\text{nor}}(\Lambda^{\text{all}}) \le 2\omega.$$

(b) *For the class* Λ^{std}*, we have*

$$\text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{std}}) \le \left\lceil 32(\rho_1+\rho_2)^4 N_{\omega/2}^4 \Gamma^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{4\omega} \right\rceil + 1.$$

Moreover, the algorithm $U_{d,n}$*, with* $n = \text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{std}})$*, gives an* ε*-approximation. Hence*

$$p_{\text{err}}^{\text{nor}}(\Lambda^{\text{std}}) \le 4 \qquad \textit{and} \qquad p_{\text{dim}}^{\text{nor}}(\Lambda^{\text{std}}) \le 4\omega.$$

Proof. Using (18), (22), and Lemma 6.1, we find that

$$C_d \le \max\left\{1, \rho_1 T\sqrt{2\max\{1,\kappa_0^{\omega}\}\gamma_{\max}}\right\} \cdot d^{\omega/2}.$$

Hence setting $\alpha = \omega/2$ in (21), we obtain (26). The remaining results of this theorem now follow from [15, Theorem 7], with $\alpha = \omega$. □

Example. Suppose that K is the min kernel $K_{\min}$. Since $\kappa_0 = 1$ and $\kappa_1 = \frac{1}{2}$, we have

$$N_{\omega/2} \le \max\left\{1, \rho_1 T\sqrt{2\gamma_{\max}}\right\}\sqrt{2\gamma_{\max}}$$

from (26). Furthermore, since $\kappa_2 = \frac{1}{3} \ne 0$, we see that case 1 holds in Theorem 6.3. Hence we find that the heat equation is tractable under the normalized error criterion, with

$$p_{\text{err}}^{\text{nor}}(\Lambda^{\text{all}}) \le 2 \qquad \text{and} \qquad p_{\text{dim}}^{\text{nor}}(\Lambda^{\text{all}}) \le \omega,$$

for continuous linear information and

$$p_{\text{err}}^{\text{nor}}(\Lambda^{\text{std}}) \le 4 \qquad \text{and} \qquad p_{\text{dim}}^{\text{nor}}(\Lambda^{\text{std}}) \le 2\omega$$

for standard information. □

Example. Now suppose that K is the Korobov kernel K_{Kor}. Since $\kappa_0 = \kappa_1 = \frac{1}{2}$, we again have

$$N_{\omega/2} \le \max\left\{1, \rho_1 T\sqrt{2\gamma_{\max}}\right\}\sqrt{2\gamma_{\max}}$$

from (26). Furthermore, since $\kappa_2 = 0$, we see that case 2 holds in Theorem 6.3, with $\Gamma \le 12$ by (25). Hence we find that the heat equation is tractable under the normalized error criterion, with

$$p^{\text{nor}}_{\text{err}}(\Lambda^{\text{all}}) \le 2 \qquad \text{and} \qquad p^{\text{nor}}_{\text{dim}}(\Lambda^{\text{all}}) \le 2\omega,$$

for continuous linear information and

$$p^{\text{nor}}_{\text{err}}(\Lambda^{\text{std}}) \le 4 \qquad \text{and} \qquad p^{\text{nor}}_{\text{dim}}(\Lambda^{\text{std}}) \le 4\omega$$

for standard information. □

Hence the heat equation is tractable under the normalized error criterion for any finite-order weighted RKHS, no matter what set of weights is used. As was the case for the absolute error criterion, we are unable to establish strong tractability at this level of generality, since the Lipschitz constant C_d s basically given by $\sigma_d(\kappa_0)$, which is a polynomial of degree $\omega/2$ in d. Hence we can only guarantee that $N_{\omega/2}$ is finite. It is proved in [15, Theorem 7] that strong tractability holds if $\kappa_2 > 0$ and if N_0 is finite. We can guarantee that N_0 is finite if we follow the approach taken in [15, Theorem 8].

Theorem 6.4. *Suppose that $\kappa_2 > 0$ and that ρ_3, as given by* (28)*, is finite. The heat equation for $H(K_d)$ with finite-order weights of order ω satisfying* (28) *is strongly tractable for the normalized error. More precisely, for N_0 defined by* (21)*, we have*

$$N_0 \le \max\left\{1, \rho_1\rho_3^{1/2} T \max\{1, \kappa_0^{\omega/2}\}\right\}, \tag{32}$$

and the following bounds hold:

1. *For the class Λ^{all}, we have*

$$\text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}}) \le 2(\rho_1 + \rho_2)^2 N_0^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} \left(\frac{1}{\varepsilon}\right)^2,$$

Moreover, the algorithm $U^*_{d,n}$, with $n = \text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{all}})$, gives an ε-approximation. Hence

$$p^{\text{abs}}_{\text{strong}}(\Lambda^{\text{all}}) \le 2.$$

2. *For the class Λ^{std}, we have*

$$\text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{std}}) \le \left\lceil 8(\rho_1 + \rho_2)^4 N_0^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 \right\rceil + 1.$$

Moreover, the algorithm $U_{d,n}$, with $n = \text{card}^{\text{nor}}(\varepsilon, S_d, F_d, \Lambda^{\text{std}})$, gives an ε-approximation. Hence

$$p^{\text{abs}}_{\text{strong}}(\Lambda^{\text{std}}) \le 4.$$

Proof. From (21) and Lemma 6.1, we have

$$N_0 = \sup_{d \in \mathbb{Z}^{++}} C_d = \max\{1, \rho_1 T \sigma_d(\kappa_0)\}.$$

Using (28), we see that

$$\sigma_d(\kappa_0) \le \rho_3^{1/2} \max\{1, \kappa_0^{\omega/2}\}.$$

Combining these results, we obtain (32). The desired result now follows from [15, Theorem 8]. $\square$

Example. Suppose once again that $K = K_{\min}$. Assume that (28) holds. Then the conditions of Theorem 6.4 are satisfied with

$$N_0 \le \rho_3^{1/2} \max\{1, \rho_1 \rho_3^{1/2} T\} \qquad \text{and} \qquad \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} = \left(\frac{3}{2}\right)^{\omega}.$$

Hence, the heat equation is strongly tractable under the normalized error criterion, with

$$p^{\text{abs}}_{\text{strong}}(\Lambda^{\text{all}}) \le 2 \qquad \text{and} \qquad p^{\text{abs}}_{\text{strong}}(\Lambda^{\text{all}}) \le 4. \quad \square$$

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