Rigid Formations with Leader-Follower Architecture

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***Abstract*— This paper is concerned with information structures used in rigid formations of autonomous agents that have leader-follower architecture. The focus of this paper is on sensor/network topologies to secure control of rigidity. We extend our previous approach for formations with symmetric neighbor relations to include formations with leader-follower architecture. Necessary and sufficient conditions for stably rigid directed formations are given including both cyclic and acyclic directed formations. Some useful steps for creating topologies of directed rigid formations are developed. An algorithm to determine the directions of links to create stably rigid directed formations from rigid undirected formations is presented. It is shown that k-cycles ($k \geq 3$) do not cause inconsistencies when measurements are noisy, while 2-cycles do. Simulation results are presented for (i) a rigid acyclic formation, (i) a flexible formation, and (iii) a rigid formation with cycles.**

I. Introduction

Multiagent systems have lately received considerable attention due to recent advances in computation and communication technologies (see for example [1]–[8]). In the context of this paper, agents will simply be thought of as autonomous agents including robots, underwater vehicles, microsatellites, unmanned air vehicles, ground vehicles, and sensor nodes. A *formation* is a group of agents moving in real 2- or 3-dimensional space. A formation is called *rigid* if the distance between each pair of agents does not change over time under ideal conditions. A formation is called *minimally rigid* if it loses it rigidity when any one of its links is removed from the formation. In other words, a minimally rigid formation has the minimum number of links to maintain rigidity. If a formation is rigid but not minimally rigid, then it is called a *redundantly rigid* formation. Minimally rigid formations are more energy-efficient compared to redundantly rigid formations. This paper is mainly concerned with minimally rigid formations. Sensing/communication links are used for maintaining fixed distances between agents. The interconnection structure of sensing/communication links is called *sensor/network topology*. In practice, actual agent groups cannot be expected to move exactly as a rigid formation because of sensing errors, vehicle modelling errors, etc. The ideal benchmark point formation against which the performance of an actual agent formation is to be measured is called a *reference formation*.

In reality, agents are entities with physical dimensions. For modeling purposes, agents are represented by points called *point agents*. Distances between all agent pairs can be held fixed by directly measuring distances between only some agents and keeping them at desired values. A 'distance constraint' is a requirement that a distance between two agents, depicted with d, be maintained through a sensing/communication link and some control strategy. Distance constraints are sometimes referred to as range or separation constraints. With enough distance constraints, the whole formation will be rigid, even without there being a distance constraint between every pair of agents.

Two agents connected by a sensing/communication link are called *neighbors*. There are two types of neighbor relations in rigid formations. In the first type, the neighbor relation is symmetric, i.e., if agent i senses/communicates with agent j and uses the received information (such as distances) for motion planning, so does agent j with agent i. A link with a symmetric neighbor relation is represented graphically by a straight line. In the second type, the neighbor relation is asymmetric, i.e., if agent i senses/communicates with agent j and uses the received information for motion planning, then agent j does not make use of any information received from agent i although it may sense/communicate with agent i. For example, rigid formations with a leader-follower architecture have the asymmetric neighbor relation. A link with an asymmetric neighbor relation between a leader and a follower is represented by a directed edge pointing from the follower to the leader, i.e., head is the leader and tail is the follower. Pointing direction from leader to follower is also used, e.g., see [9]. The terms *undirected formation* and *directed formation* are also used to describe formations with symmetric neighbor relations and formations with leader-follower architecture [3], respectively. We will also use those terms throughout the paper.

Eren et al. [10]–[12] and Olfati-Saber and Murray [2] suggested an approach based on rigidity for maintaining formations of autonomous agents with sensor/network topologies that use distance information between agents, where the neighbor relation is symmetric. Rigidity of undirected formations with distance information is well understood in 2-dimensional space, and there are partial results in 3-dimensional space [12]. For formations that have a leader-follower architecture, Baillieul and Suri gave two separate conditions for stable rigidity for formations that have distance information between agents, one of which is a necessary condition and the other is a sufficient condition [5]. Tanner et al. studied input-to-state stability properties of formations with cyclic interconnections in [13]. Desai et al. made use of both distance and bearing information to maintain formations that have leader-follower architecture [9]. This paper suggests an approach to analyze rigid formations with leader-follower architecture and proves that the necessary condition given by Baillieul and Suri is a necessary and sufficient condition for stable rigidity in acyclic

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directed formations.

We show that redundantly rigid formations lead to overdetermined systems. Inconsistencies in overdetermined systems caused by redundant rigidity are called *redundancy-based inconsistencies*. Although 2-cycles cause redundancy-based inconsistencies, we show that cycles of length 3 or more do not cause redundancy-based inconsistencies. We then provide necessary and sufficient conditions for stably rigid directed formations that have k-cycles ($k \geq 3$).

In this paper, we restrict our attention to minimally rigid formations in 2-dimensional space. We wish to consider a broader range of interconnection topologies, including both cyclic and acyclic, and understand how the interconnection topology and the directions affect the rigidity of a formation as it performs a coordinated motion. Our ultimate goal is the development of strategies to create minimally rigid directed formations, which are scalable for any number of agents.

The contributions of this paper are:

1) to extend our previous approach for formations with symmetric neighbor relations to include formations with leader-follower architecture;
2) to give necessary and sufficient conditions for stably rigid directed formations including both cyclic and acyclic directed formations;
3) to develop some useful steps for creating sensor/network topologies of directed rigid formations;
4) to present a procedure to determine the directions of links to create a stably rigid formation from a rigid undirected formation;
5) to show that k-cycles ($k \geq 3$) do not cause inconsistencies when measurements are noisy, while 2-cycles do;
6) to show that redundant rigidity is a source for inconsistencies when measurements are noisy.

The paper is organized as follows. In §II, we start with definitions of rigidity. We review point formations in §II-A, and rigid formations with symmetric neighbor relations in §II-B. We investigate rigid formations that have leader-follower architecture in §III. Cycles in rigid formations are studied in §IV. We focus on creating directed rigid formations from undirected rigid formations in §V. Finally, concluding remarks are given in §VI.

II. Rigidity and Point Formations

One way of visualizing rigidity with symmetric neighbor relation is to imagine a collection of rigid bars connected to one another by idealized ball joints, which is called a bar-joint framework. By an idealized ball joint we mean a connection between a collection of bars which imposes only the restriction that the bars share common endpoints. Now, can the bars and joints be moved in a continuous manner without changing the lengths of any of the bars, where translations and rotations do not count? If so, the framework is flexible; if not, it is rigid. (Precise definitions will appear in the sequel.) In a bar-joint framework, the length of a bar imposes a distance constraint for both end-joints. This is the same situation in a formation where two agents connected by a sensing/communication link are mutually affected by the information conveyed by this link. For example, if two agents connected by a sensing/communication link are set to maintain a ten meter distance between each other, then both agents perform action to maintain this distance. In the graph theoretic setting, the edge corresponding to this link is denoted by an undirected edge.

The situation in a rigid formation where the relation between agents has a leader-follower architecture is different, because the information on a sensing/communication link between a leader-follower pair is used only by the follower. For example, with the same distance requirement as in the example above, if two agents, labelled with i and j, are set to maintain a ten meter distance between themselves where i is the leader and j is the follower, then only agent j performs action to maintain this distance. Let us assume the following properties in a formation of agents: (i) there is a global formation leader that determines where the entire formation moves, and it does not follow any other member; (ii) there is a first-follower of the global leader that maintains a predefined distance only to the global leader; (iii) every other agent of the formation maintains predefined distances to some other agents in the formation; (iv) if an agent, say B, maintains a predefined distance to another agent, say A, then A does not perform any action to maintain a predefined distance to B (in this relation A is a leader and B is a follower). As the formation moves with the leadership of the global leader, if the distance between every pair of agents does not change over time under ideal conditions, then such a formation is a rigid formation.

Certain directed information patterns in a formation can be described by bar-joint frameworks. To do that, consider creating a bar-joint framework in the plane starting from two joints connected by a bar. Once the end-joints are held fixed (i.e., translations and rotations are avoided), we can insert a new joint by connecting it to the existing joints using new bars. In this scenario, the constraints imposed by the new bars act only on the newly inserted joint because the initial bar-joint framework is already fixed and cannot be affected by the newly inserted bars and joints. (If the first two joints are regarded as agents i and j, and the new joint is regarded as agent k, then agent k performs the actions to maintain its distance from i and j, while i and j do not perform any corresponding action in relation to agent k.) If the resulting bar-joint framework is not deformable, then this new resulting bar-joint framework is rigid and it becomes the new fixed bar-joint framework for the next step. In the graph-theoretic setting, the directed edge points to the newly inserted joint from the fixed bar-joint framework.

To summarize, there are two types of neighbor relations. They can be symmetric, i.e., if agent i senses/communicates with agent j and performs action upon the information it receives, so does agent j. This corresponds to an undirected graph. Alternatively, the formation can have a leader-follower architecture, i.e., agent j senses/communicates with agent i and performs actions upon the information it receives, but the actions of agent i do not depend on the information conveyed by the sensing/communication with agent j. The underlying graph of such formations is a directed graph. A directed edge points from the leader to the follower. We will consider these two cases separately in each section.

A. Point Formations

A point formation $\mathbb{F}_p \triangleq (p, \mathcal{L})$ provides a way of representing a formation of n agents. $p \triangleq \{p_1, p_2, \ldots, p_n\}$ and the points p_i represent the positions of agents in $\mathbb{R}^2$ where i is an integer in $\{1, 2, \ldots, n\}$ and denotes the labels of agents. $\mathcal{L}$ is the set of "maintenance links," labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$. The *maintenance links* in $\mathcal{L}$ correspond to constraints between specific agents, such as distances, which are to be maintained over time by using sensing/communication links between certain pairs of agents.

Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G}_{\mathbb{F}_p} \triangleq (\mathcal{V}, \mathcal{L})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$, which is the set of labels of agents, and edge set $\mathcal{L}$. A formation with distance constraints can be represented by $(\mathcal{V}, \mathcal{L}, f)$ where $f : \mathcal{L} \longmapsto \mathbb{R}$. Each maintenance link $(i, j) \in \mathcal{L}$ is used to maintain the distance $f((i, j))$ between certain pairs of agents fixed.

A *trajectory* of a formation is a continuously parameterized one-parameter family of curves $(q_1(t), q_2(t), ..., q_n(t))$ in $\mathbb{R}^{nd}$ which contain p and on which for each t, $\mathbb{F}_{q(t)}$ is a formation with the same measured values under f. A *rigid motion* is a trajectory along which point formations contained in this trajectory are congruent to each other. We will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_r$, where $p, r \in q(t)$, are congruent if they have the same graph and if p and r are congruent. p is *congruent* to r in the sense that there is a distance-preserving map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(r_i) = p_i, i \in \{1, 2, \ldots, n\}$. If rigid motions are the only possible trajectories then the formation is called *rigid*; otherwise it is called *flexible* [10].

B. Rigidity in Point Formations with Symmetric Neighbor Relations

Whether a given point formation is rigid or not can be studied by examining what happens to the given point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ with m maintenance links, along the trajectory $q([0, \infty)) \triangleq \{\{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ on which the Euclidean distances $d_{ij} \triangleq ||p_i - p_j||$ between pairs of points (p_i, p_j) for which (i, j) is a link are constant. Along such a trajectory

$$(q_i - q_j) \cdot (q_i - q_j) = d_{ij}^2, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \tag{1}$$

We note that the existence of a trajectory is equivalent to the existence of a piecewise analytic path, with all derivatives at the initial point [14]. Assuming a smooth (piecewise analytic) trajectory, we can differentiate to get

$$(q_i - q_j) \cdot (\dot{q}_i - \dot{q}_j) = 0, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \tag{2}$$

Here, $\dot{q}_i$ is the velocity of point i. The m equations can be collected into a single matrix equation

$$R_{\mathcal{L}}(q)\dot{q} = 0 \tag{3}$$

where $\dot{q} = [\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n]^T$ and $R_{\mathcal{L}}(q)$ is a specially structured $m \times 2n$ matrix called the *rigidity matrix* [15]–[17].

Example 2.1: Consider a planar point formation $\mathbb{F}_p$ shown in Figure 1. This has a rigidity matrix as shown in Table I.

Let $\mathcal{M}_p$ be the manifold of points congruent to p. Because any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$, is one along which $\mathbb{F}_p$ undergoes rigid motion, (2) automatically holds along any trajectory which lies within $\mathcal{M}_p$. From this, it follows that the tangent space to $\mathcal{M}_p$ at p, written $\mathcal{T}_p$, must be contained in the kernel of $R_{\mathcal{L}}(p)$. If the points $p_1, p_2, \ldots, p_n$ are in general position (which means that the points $p_1, p_2, \ldots, p_n$ do not lie on any hyperplane in $\mathbb{R}^n$), then $\mathcal{M}_p$ is $n(n+1)/2$ dimensional since it arises from the $n(n-1)/2$-dimensional manifold of orthogonal transformations of $\mathbb{R}^n$ and the n-dimensional manifold of translations of $\mathbb{R}^n$ [15]. Thus $\mathcal{M}_p$ is 3-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^2$. We have $\text{rank } R_{\mathcal{L}}(p) = 2n - \text{dimension}\{\text{kernel}(R_{\mathcal{L}}(p))\} \leq 2n - n(n+1)/2$. The following theorem holds [15], [16]:

Theorem 2.2: Assume $\mathbb{F}_p$ is an n-point formation with at least 2 points in 2-dimensional space where $\text{rank } R_{\mathcal{L}}(p) = \max\{\text{rank } R_{\mathcal{L}}(x) : x \in \mathbb{R}^2\}$. $\mathbb{F}_p$ is rigid in $\mathbb{R}^2$ if and only if

$$\text{rank } R_{\mathcal{L}}(p) = 2n - 3.$$

This theorem leads to the notion of the "generic" behavior of rigidity. When the rank is less than the maximum, the formation may still be rigid. However this type of rigidity lacks the generic behavior and thus is not addressed in this paper.

1) Generic Rigidity: We define a type of rigidity, called "generic rigidity," that is more useful for our purposes. A set $\mathcal{A} = (\alpha_1, \ldots, \alpha_m)$ of distinct real numbers is said to be *algebraically dependent* if there is a non-zero polynomial $h(x_1, \ldots, x_m)$ with integer coefficients such that $h(\alpha_1, \ldots, \alpha_m) = 0$. If $\mathcal{A}$ is not algebraically dependent, it is called *generic* [18]. We say that $p = (p_1, \ldots, p_n)$ is generic in 2-dimensional space, if its $2n$ coordinates are generic. It can be shown that the set of generic p's form an open connected dense subset of $\mathbb{R}^{2n}$ [19]. A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is called *generically rigid*, if $\mathbb{F}_p = (p, \mathcal{L})$ is rigid for a generic p.

The concept of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}_p$ but examines how well the rigidity of formations can be judged by knowing the vertices and their incidences, in other words, by knowing the underlying graph. For this reason, it is a desirable specialization of the concept of a "rigid formation" for our purposes. The following theorem holds for a generically rigid graph [16]:

Theorem 2.3: The following are equivalent:

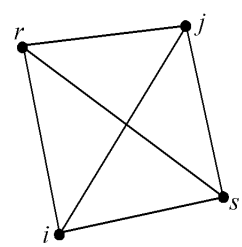

Fig. 1. A planar point formation used to demonstrate the rigidity matrix. The rigidity matrix corresponding to this point formation is shown in Table 2.1.

$R_{\mathcal{L}}(p)$	i		j		r		s	
(i,j)	$x_i - x_j$	$y_i - y_j$	$x_j - x_i$	$y_j - y_i$	0	0	0	0
(i,r)	$x_i - x_r$	$y_i - y_r$	0	0	$x_r - x_i$	$y_r - y_i$	0	0
(i,s)	$x_i - x_s$	$y_i - y_s$	0	0	0	0	$x_s - x_i$	$y_s - y_i$
(j,r)	0	0	$x_j - x_r$	$y_j - y_r$	$x_r - x_j$	$y_r - y_j$	0	0
(j,s)	0	0	$x_j - x_s$	$y_j - y_s$	0	0	$x_s - x_j$	$y_s - y_j$
(r,s)	0	0	0	0	$x_r - x_s$	$y_r - y_s$	$x_s - x_r$	$y_s - y_r$

TABLE I
RIGIDITY MATRIX EXAMPLE FOR DISTANCES

1) a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in 2-dimensional space;
2) for some p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ has rank$\{R_{\mathcal{L}}(p)\}$= $2|\mathcal{V}| - 3$ where $|\mathcal{V}|$ denotes the cardinal number of $\mathcal{V}$;
3) for almost all p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ is rigid.

For 2-dimensional space, we have a complete combinatorial characterization of generically rigid graphs, which was first proved by Laman in 1970 [20].

Theorem 2.4 (Laman [20]): A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in 2-dimensional space if and only if there is a subset $\mathcal{L}' \subseteq \mathcal{L}$ satisfying the following two conditions: (1) $|\mathcal{L}'| = 2|\mathcal{V}| - 3$, (2) For all $\mathcal{L}'' \subseteq \mathcal{L}', \mathcal{L}'' \neq \emptyset, |\mathcal{L}''| \leq 2|\mathcal{V}(\mathcal{L}'')| - 3$, where $|\mathcal{V}(\mathcal{L}'')|$ is the number of vertices that are end-vertices of the edges in $\mathcal{L}''$.

2) Sequential Techniques: In this section, we present sequential techniques to create minimally rigid point formations. As noted earlier, Laman's Theorem characterizes rigidity in 2-dimensional space. There are sequential techniques for generating rigid classes of graphs in 2-dimensional space based on what are known as the vertex addition, edge splitting, and vertex splitting operations. First, we introduce the first two of these three operations, namely the vertex addition and edge splitting operations. Then we present sequences to create rigid point formations in which these operations are used. Before explaining these operations and sequences, we introduce some additional terminology. We shall omit discussion here of vertex splitting.

If (i,j) is an edge, then we say that i and j are *adjacent* or that j is a *neighbor* of i and i is a neighbor of j. The vertices i and j are *incident* with the edge (i,j). Two edges are *adjacent* if they have exactly one common end-vertex. The *degree* or *valency* of a vertex i is the number of neighbors of i. If a vertex has k neighbors, it is called a *vertex of degree* k or a k*-valent vertex*. The set of neighbors of i, denoted by $\mathcal{N}_{\mathbb{G}}(i)$, is called a (open) neighborhood. When i is also included, it is called a closed neighborhood and is denoted by $\mathcal{N}_{\mathbb{G}}[i]$. The subscript $\mathbb{G}$ is usually dropped when there is no danger of confusion.

One graph expansion operation is *vertex addition*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$, we add a new vertex i with two edges between i and two other vertices in $\mathcal{V}^*$ in 2-dimensional space. A second operation is *edge splitting*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$, we remove an edge (j,k) in $\mathcal{L}^*$ and then we add a new vertex i with three edges by inserting two edges (i,j), (i,k) and one edge between i and one vertex (other than j,k) in $\mathcal{V}^*$.

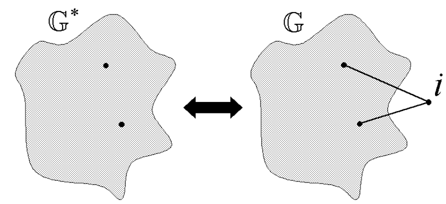

Fig. 2. Vertex addition in 2-dimensional space - undirected case.

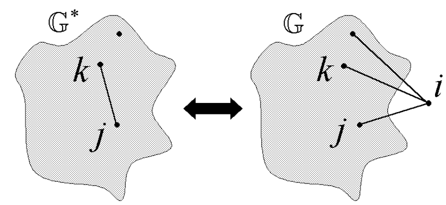

Fig. 3. Edge splitting in 2-dimensional space - undirected case.

Now we are ready to present the following theorems:

Theorem 2.5: (vertex addition in undirected case - Tay, Whiteley [21]) Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree 2 in 2-dimensional space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ denote the subgraph obtained by deleting i and the edges incident with it. Then $\mathbb{G}$ is generically minimally rigid if and only if $\mathbb{G}^*$ is generically minimally rigid.

Example 2.6: Vertex addition in 2-dimensional space for an undirected graph is shown in Figure 2.

Theorem 2.7: (edge splitting in undirected case - Tay, Whiteley [21]) Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree 3, and let $\mathbb{G}' = (\mathcal{V}', \mathcal{L}')$ be the subgraph obtained by deleting i and its three incident edges. Then $\mathbb{G}$ is generically minimally rigid if and only if there is a pair j,k of the neighborhood $\mathcal{N}_{\mathbb{G}}(i)$ such that the edge (j,k) is not in $\mathcal{L}$ and the graph $\mathbb{G}^* = (\mathcal{V}', \mathcal{L}' \bigcup (j,k))$ is generically minimally rigid.

Example 2.8: Edge splitting in 2-dimensional space for an undirected graph is shown in Figure 3.

Vertex addition and edge splitting operations are used in Henneberg sequences.

3) Henneberg Sequences: Henneberg sequences are a systematic way of generating minimally rigid graphs based on the vertex addition and edge splitting operations [21]. In 2-dimensional space, we are given a sequence of graphs: $\mathbb{G}_2, \mathbb{G}_3, \ldots, \mathbb{G}_{|\mathcal{V}|}$ such that:

1) $\mathbb{G}_2$ is the complete graph on two vertices;
2) $\mathbb{G}_{i+1}$ comes from $\mathbb{G}_i$ by adding a new vertex either by i) the vertex addition or ii) the edge splitting operation, where $\mathbb{G}_i$ has i vertices.

Note that $\mathbb{G}_i$ and $\mathbb{G}_{i+1}$ correspond to $\mathbb{G}^*$ and $\mathbb{G}$ in the statements of Theorem 2.5 and Theorem 2.7. All graphs in the sequence are minimally rigid in 2-dimensional space.

Theorem 2.9 (Henneberg's Theorem [16]): A graph $\mathbb{G}$ with at least two vertices is minimally rigid if and only if $\mathbb{G}$ has a Henneberg sequence.

III. Rigidity in Directed Formations

First, we give some definitions from graph theory, which are relevant to point formations with leader-follower architecture. A graph in which each edge is replaced by a directed edge is called a *digraph*, also called a *directed graph*. When there is a danger of confusion, we will call a graph, which is not a digraph, an *undirected graph*. A digraph having no multiple edges or loops (corresponding to a binary adjacency matrix with 0's on the diagonal) is called a *simple digraph*.

An *arc*, or *directed edge*, is an ordered pair of end-vertices. It can be thought of as an edge associated with a direction. Each directed edge is denoted with a line directed from the first element to the second element of the pair. For example, for a given directed edge (i, j), the direction is from i to j. Symmetric pairs of directed edges are called *bidirected edges*. In the context of formations, a birected edge is equivalent to an undirected edge in the underlying graph of a formation. We will use only directed graphs with no bidirected edges in formations that have a leader-follower architecture. The number of inward directed graph edges to a given graph vertex i in a directed graph $\mathbb{G}$ is called the *in-degree* of the vertex and is denoted by $d^-_{\mathbb{G}}(i)$. The number of outward directed graph edges from a given graph vertex i in a directed graph $\mathbb{G}$ is called the *out-degree* of the vertex and is denoted by $d^+_{\mathbb{G}}(i)$. The set of neighbors of i such that the directed edge is pointed from i to the other vertex, denoted by $\mathcal{N}_{\mathbb{G}}(i)$, is called a (open) neighborhood. When i is also included, it is called a closed neighborhood and is denoted by $\mathcal{N}_{\mathbb{G}}[i]$. The out-neighborhood $N^+_{\mathbb{G}}(i)$ of a vertex i is $\{j \in \mathcal{V} : (i, j) \in \mathcal{L}\}$, and the in-neighborhood $N^-_{\mathbb{G}}(i)$ of a vertex i is $\{j \in \mathcal{V} : (j, i) \in \mathcal{L}\}$. A *path* is a sequence $\{i, j, k, \dots, r, s\}$ such that (i, j), (j, k), $\dots$, (r, s) are edges of the graph. A *cycle* of a graph $\mathbb{G}$ is a subset of the edge set of $\mathbb{G}$ that forms a path such that the first vertex of the path corresponds to the last. This definition usually refers to what is known as a circuit, or closed walk. When stated without any qualification, a cycle of n vertices, denoted by $\mathcal{C}_n$, is usually assumed to be a simple cycle, meaning every vertex is incident to exactly two edges. The *length of a cycle* is the number of its edges. Cycles of length 1 are loops. Cycles of length 2 are pairs of multiple edges. We call a cycle of k edges a *k-cycle*. A $k-$cycle is represented by an n-tuple consisting of its vertices separated by commas, e.g., (i, j, k, i). A *directed cycle* is an oriented cycle such that all arcs go the same direction. A digraph is *acyclic* if it does not contain any directed cycle.

In a formation with leader-follower architecture, each link is denoted with a line directed from follower to leader. One type of topology is as follows: There is one global leader and one first-follower of the global leader. The global leader does not follow any other agent, and the first-follower only follows the global leader. They are connected with one link pointed from the first-follower to the global leader. The rest of the agents are followers of at least two other agents. They can also be leaders of other agents. Figure 4(a) shows such an architecture.

It is straightforward to see that ordinary agents (agents other than the global leader and the first follower) have at least two links. The global leader has 2 degrees of freedom, the first follower has 1 degree of freedom, which makes 3 degrees of freedom in total. This allows them to control translation and rotation of a formation. If any one of ordinary agents has less than two links, this results in an additional degree of freedom. Then the formation cannot maintain rigidity anymore. Possible selections of directions of links and the number of out-going links from points are implicit in the paper by White and Whiteley [22]. A an algorithm is given in detail in §V.

One can also consider other types of topologies. For example, the first-follower follows only one agent, but not necessarily the global leader. There will be still one global leader of out-degree 0, one first follower of out-degree 1, and other agents of out-degree 2. Figure 4(b) shows such an example. Another possibility is that all agents have out-degree 2, except three agents of out-degree 1 as shown in Figure 4(c).

We will focus on the topology where the global leader of out-degree 0, and the first follower of out-degree 1 are neighbors. This will ensure that the global leader and the first follower are not part of a cycle. Thus the translation and rotation of formation are controlled by the global leader's and the first follower's actions, including the measurement and actuation errors they make, but not affected by measurement or actuation errors that other agents make.

In a rigid formation with leader-follower architecture, once we fix the positions of the global leader and the first-follower, the formation cannot deform, including translations and rotations. The global leader and the first-follower can make the entire rigid formation translate and rotate in 2-dimensional space by making maneuvers.

Recall that the global leader has no out-going links and the first follower has one link of out-degree 1. Since every other agent has at least two links with an out-degree of 2, we expect at least $2(n-2)+1 = 2n-3$ links in total.

For point formations with leader-follower architecture, Baillieul and Suri define stably rigid formations [5]. They first introduce a general model for distributed relative distance control of a point formation:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \sum_{j \in \mathcal{N}^+_{\mathbb{G}}(i)} u_{ij}\left(d_{ij}, \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\right) \begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} \quad (4)$$

for $i \neq 1, 2$ where d_{ij} is the set-point distance between agents i and j, and u_{ij} is a function of both the set-point and the measured distance. The definition of stable rigidity is as follows: a formation is *stably rigid* under a distributed relative distance control law as given in (4), if for any sufficiently small perturbation in the relative positions of the agents, the

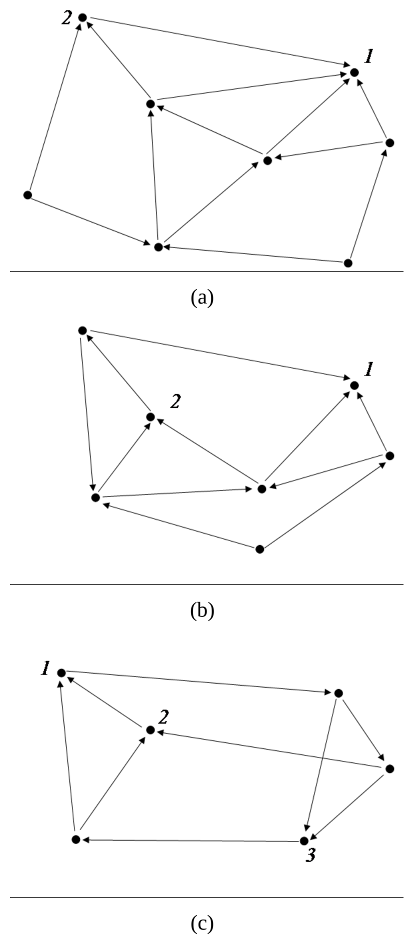

Fig. 4. Three different topologies for a leader-follower architecture are shown. In (a), all vertices are of out-degree 2, except that there is one vertex of out-degree 0 (labeled with 1), and another vertex of out-degree 1 (labeled with 2), and these two vertices are neighbors. In (b), all vertices are of out-degree 2, except that there is one vertex of out-degree 0 (labeled with 1), and another vertex of out-degree 1 (labeled with 2), and these two vertices are not neighbors. In (c), all vertices are of out-degree 2, except that there are three vertices of out-degree 1. These vertices are labeled with 1, 2, and 3.

control law steers them asymptotically back into the prescribed formation in which the relative distance constraints are satisfied. Notice that this requirement justifies why we avoid the topologies in which all agents have out-degree 2, except three agents of out-degree 1 such as $\mathcal{C}_3$ and the topology shown in Figure 4(c). Any small perturbation in agents' positions causes such a formation to move from a desired position. The following theorem is given in [5] as a sufficiency condition for stably rigid formations:

Theorem 3.1 (Baillieul and Suri): If a formation is constructed from a single directed edge by a sequence of vertex addition operation, then it is stably rigid.

The following proposition is given in [5] as a necessary condition for stable rigidity:

Proposition 3.2 (Baillieul and Suri): If a formation with directed links is stably rigid then the following three conditions hold for the underlying graph: i) the undirected underlying graph is generically rigid; ii) the directed graph is acyclic; iii) the directed graph has no vertex with an out-degree greater than 2.

For the time being, we assume that acyclicity is a necessary condition for stable rigidity as it is given in Proposition 3.2. In §IV, we will show that k-cycles where $k \geq 3$ do not cause instability. We will give the necessary and sufficient conditions for stable rigidity for formations that have cycles in §IV. In this section, we focus on acyclic graphs. It is stated in [5] that the conditions in Proposition 3.2 are not sufficient because there is a counterexample graph shown in Figure 5, i.e., this graph satisfies the conditions of Proposition 3.2 but it is not stably rigid. However, we note that this graph actually does not satisfy the conditions of Proposition 3.2, because there is a cycle $(3, 5, 4, 6, 3)$ in the graph; hence it violates the condition ii) of Proposition 3.2. It can be proved that the conditions given in Proposition 3.2 are also sufficient conditions; hence these conditions are necessary and sufficient conditions for stable rigidity. Minimal rigidity together with acyclicity in a directed graph implies all vertices have out-degree at most 2. Therefore, the third condition in Proposition 3.2 is redundant. We have the following proposition:

Proposition 3.3: A point formation in 2-dimensional space with directed links is stably rigid if and only if the following conditions hold for the underlying directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$: i) the undirected graph is generically minimally rigid; ii) the directed graph is acyclic.

Proof: The necessity part of the proof is proved in [5]. Here we prove the sufficiency part only. Let us assume that the directed graph is acyclic. Then we can take the directed edges to define a partial order on the vertices: $a \geq b$ if the directed edge is pointed from a to b. We can extend this by transitivity. Since there are no cycles, this is a partial order with all vertices distinct. Since the graph is minimally generically rigid, all vertices have degree at least 2. Any maximal elements in this partial order have only outgoing edges - and therefore has two such edges. This can be removed (by the reversed vertex addition operation) to give a smaller, minimally rigid graph satisfying all of the conditions. We continue this down to one directed edge. The end points of this directed edge become the global leader and the first follower. Since this reduction sequence can be reversed, the graph is constructed using only the vertex addition operation. By Theorem 3.1, such graphs are stably rigid. □

Corollary 3.4: Equivalently a point formation in 2-dimensional space that has a leader-follower architecture is stably rigid if and only if the point formation can be constructed from the initial edge by the vertex addition operation.

Figure 6 shows a formation created by vertex addition only. Notice that this formation satisfies the conditions in Proposition 3.3. The agent with out-degree 0 (global leader) is labeled with 1 and the agent with out-degree 1 is labeled with 2. Agents are fully actuated omnidirectional point agents, i.e. they can move in any direction with any speed. The trajectories of agents obtained in simulations are shown in Figure 7. As the global leader moves on a zigzag trajectory, rigidity is preserved as shown in Figure 8.

In simulations throughout the paper the following distributed relative distance control is used:

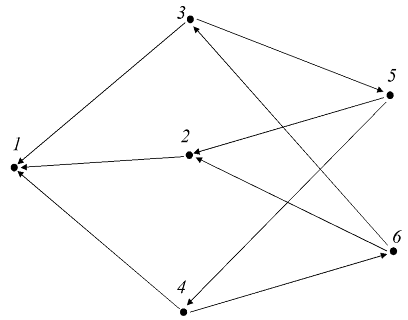

Fig. 5. This is the digraph given in [5] as a counterexample to the sufficiency of Proposition 3.2. Notice that there exist cycles in this digraph, e.g., $(3, 5, 4, 6, 3)$, so it actually does not serve as a counterexample.

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \sum_{j \in \mathcal{N}^{+}_{\mathbb{G}}(i)} \left[d_{ij} - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \right] \begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} \quad (5)$$

for $i \neq 1, 2$ where d_{ij} is the set-point distance between agents i and j. The distances between all agent pairs remain (almost) constant over time as the formation moves. Small deviations from constant link lengths are due to actuation errors that are allowed intentionally. Notice that the control law given in (5) requires infinite amount of time to reach the desired position, so small actuation errors are allowed. In simulations throughout the paper, measurement noise levels on link lengths are randomly chosen at the beginning of the program and remain constant over time. Noise level ranges between 0% and 15% of link lengths.

Recall that the global leader and the first follower control the translation and rotation of the formation. In this simulation, only the global leader's trajectory is prescribed to control translation. Intentionally, we did not prescribe the one degree of freedom that the first follower had, so that more challenging trajectories are generated to test rigidity. When we prescribe the trajectories of both the global leader and the first follower such that rotations are disallowed, then we obtain translation-only motion.

Figure 9 shows a formation that does not satisfy the conditions given in Proposition 3.3. In particular, it fails the condition i). The agent of out-degree 0 (global leader) is colored in red and the agents of out-degree 1 are colored with green. The agents of out-degree 2 are colored in blue. As the global leader moves, the rigidity is lost. This can be seen in Figure 11. The distances between agent pairs, where there is no sensing/communication link, change over time as the formation moves.

The edge split operation is not used in [5] because this operation, as described in [5], results in vertices of out-degree 3. However, the edge split operation can be defined in such a way that the out-degrees of vertices remain less than 3. The definition given below for the edge split operation on directed minimally rigid graphs results vertices of out-degree 2.

4) Sequential Techniques: As with undirected graphs, one operation for graph expansion is *vertex addition*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$, we add a new vertex i

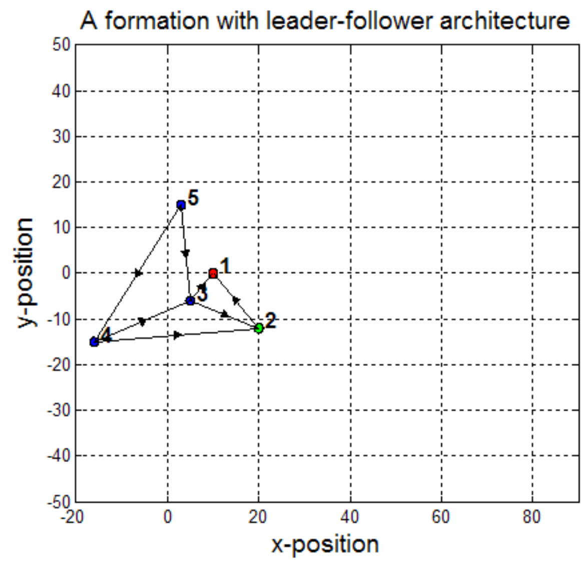

Fig. 6. A rigid formation created by vertex addition only. It satisfies the conditions given in 3.3. The agent of out-degree 0 (global leader) is depicted with color red and has index 1. The agent of out-degree 1 is depicted with green and has index 2. The agents of out-degree 2 are depicted with color blue and has indices 3, 4, 5.

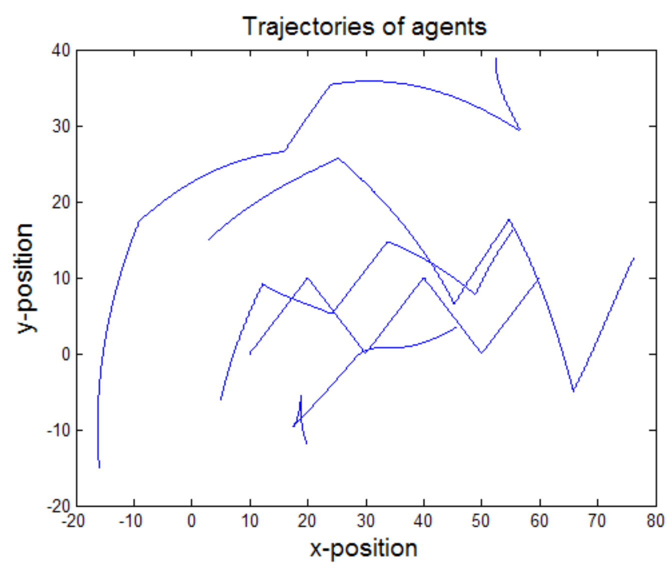

Fig. 7. Trajectories of agents in the formation shown in Figure 6.

of out-degree 2 with two edges directed from i to two other vertices in $\mathcal{V}^*$. The second operation is *edge splitting*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$, we remove a directed edge (j, k) (directed from j to k) in $\mathcal{L}^*$ and then we add a new vertex i of out-degree 2 and in-degree 1 with three edges by inserting two edges (j, i), (i, k), and one edge between i and one other vertex (other than j, k) in $\mathcal{V}^*$ such that the edge (j, i) is directed from j to i and the other two edges are directed form i to the other vertices.

Now we are ready to present the following theorems:

Theorem 3.5 (vertex addition - directed case): Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a directed graph with a vertex i of out-degree 2 in 2-dimensional space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ denote the subgraph obtained by removing i and the edges incident with it. Then

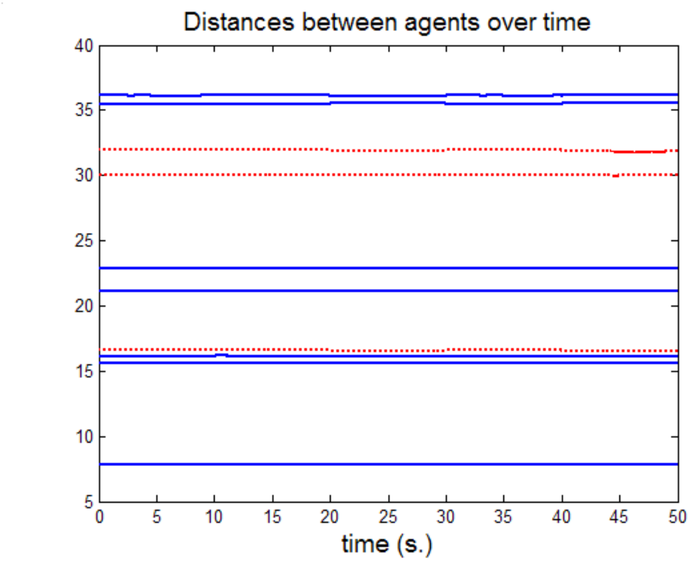

Fig. 8. Distances between agent pairs (of the formation shown in Figure 6) are shown over time as the global leader moves. The blue solid lines show the distances between agents where there exist links between those pairs. The red dotted lines show the distances between agent pairs where there are no links between those pairs.

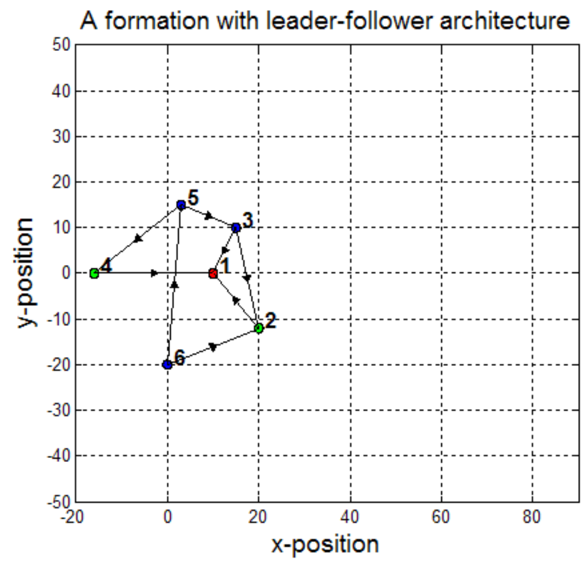

Fig. 9. A flexible formation is shown. It does not satisfy the conditions given in 3.3, i.e., the underlying undirected graph is flexible . The agent of out-degree 0 (global leader) is depicted with color red and has index 1. The agents of out-degree 1 are depicted with green and have indices 2 and 4. The agents of out-degree 2 are depicted with color blue and have indices 3, 5, and 6.

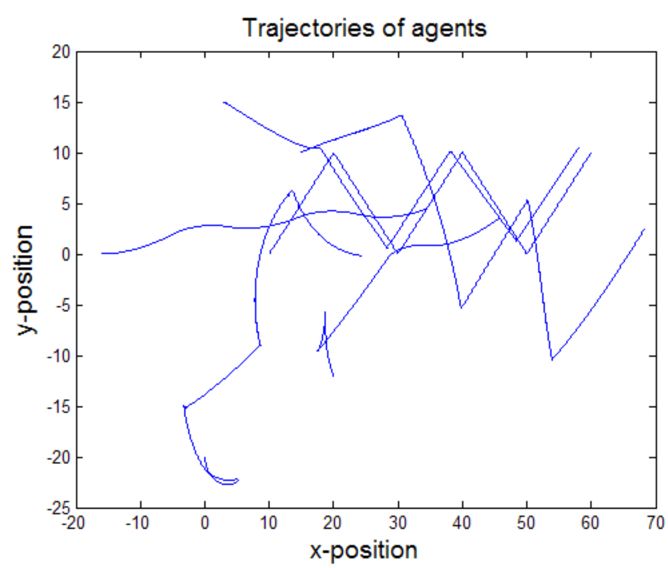

Fig. 10. Trajectories of agents in the formation shown in Figure 9.

$\mathbb{G}$ is stably rigid if and only if $\mathbb{G}^*$ is stably rigid.

Proof: Inserting/removing i from the undirected graph $\mathbb{G}$ is equivalent to the vertex addition operation in an undirected graph. Undirected minimally rigid graphs maintain rigidity under the vertex addition operation. Hence condition i) in Proposition 3.2 is satisfied in both $\mathbb{G}$ and $\mathbb{G}^*$.

If $\mathbb{G}^*$ is acyclic (condition ii) in Proposition 3.2), adding a new vertex i of out-degree 2 does not add a cycle to the resulting graph $\mathbb{G}$. This is because the inserted new vertex has only out-going edges, hence no path can come and leave that vertex. If $\mathbb{G}$ is acyclic, then the graph resulting from the removal of a vertex and its incident edges is also acyclic because removal of vertices and edges do not create new paths.

Now suppose that $\mathbb{G}$ satisfies the condition iii) in Proposition 3.2. If we remove i, then the out-degrees of the vertices of $\mathbb{G}^*$ do not change. Similarly, suppose that $\mathbb{G}^*$ satisfies the condition iii). If insert i with out-degree 2, then the out-degrees of the remaining vertices do not change. □

Example 3.6: The vertex addition operation for a directed graph is shown in Figure 12.

Theorem 3.7 (edge splitting - directed case): Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of out-degree 2 and in-degree 1 (where this edge is between i and j), and let $\mathbb{G}' = (\mathcal{V}', \mathcal{L}')$ be the subgraph obtained by deleting i and its three incident edges. Then $\mathbb{G}$ is stably rigid if and only if there is a directed edge of a pair j, k (directed from j to k) of the neighborhood $\mathcal{N}_{\mathbb{G}}(i)$ such that the directed edge (j, k) is not in $\mathcal{L}$ and the graph $\mathbb{G}^* = (\mathcal{V}', \mathcal{L}' \bigcup (j, k))$ is stably rigid.

Proof: Condition i) in Proposition 3.2 is the edge splitting operation for undirected graphs as explained in §II-B.

It is straightforward to see that if $\mathbb{G}$ is acyclic, i.e. it satisfies condition ii) in Proposition 3.2, then $\mathbb{G}^*$ is acyclic because removing a vertex and its incident edges from an acyclic graph does not add any new paths to the graph and hence does not add any cycles. Now suppose that $\mathbb{G}^*$ is acyclic. If a new vertex i is added as described in the edge splitting operation, by removing the edge (j, k) and inserting the edges (j, i), (i, k), (i, s) (where $s \in \mathcal{V}'$), then there is a possibility of the existence of a cycle that goes through the edges (j, i) and (i, s). An example of such a possibility is depicted in Figure 13. However, if one of the new edges is inserted between the new vertex and the first follower (directed from the new vertex to the first follower) then a cycle can always be avoided.

Suppose that $\mathbb{G}$ satisfies condition iii) in Proposition 3.2. Then removing i only changes the out-degree of j. However, an edge is inserted from j to k. So all the vertices of $\mathbb{G}^*$ have out-degree of 2. Conversely, suppose that $\mathbb{G}^*$ satisfies

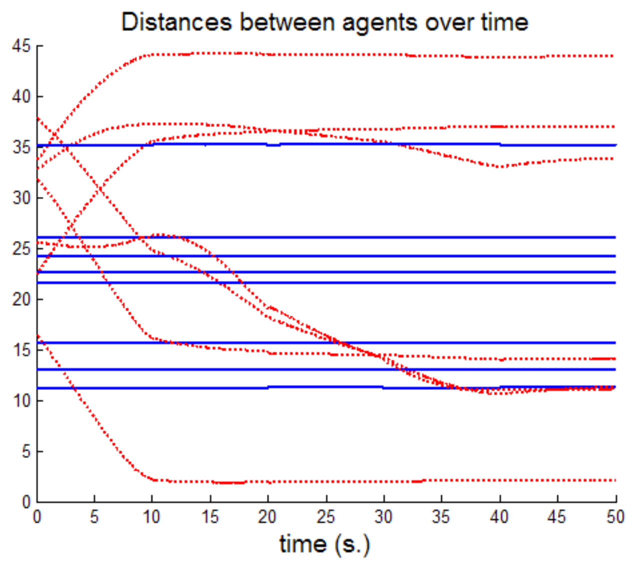

Fig. 11. Distances between agent pairs (of the formation shown in Figure 9) are shown over time as the global leader moves. The blue solid lines show the distances between agents where there exist links between those pairs. The red dotted lines show the distances between agent pairs where there are no links between those pairs.

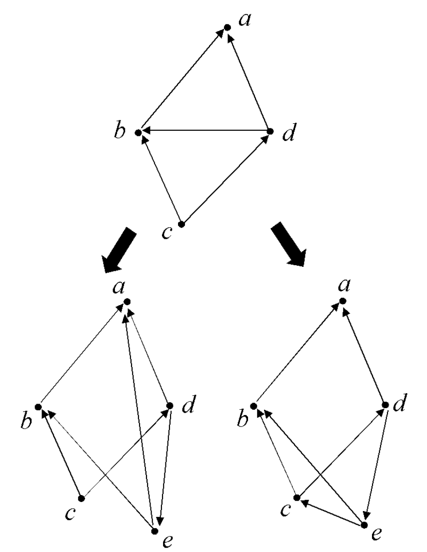

Fig. 13. Two examples of the edge splitting operation on a minimally rigid directed graph. The split edge is (d, b). We note that the resulting directed graph on the left has no cycles. On the other hand, the resulting directed graph on the right has a cycle (c, d, e, c). We note that the acyclic directed graph on the left can also be obtained by a series of vertex additions starting from a single edge.

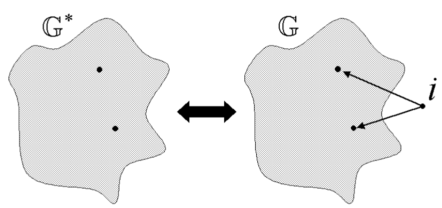

Fig. 12. Vertex Addition - directed case.

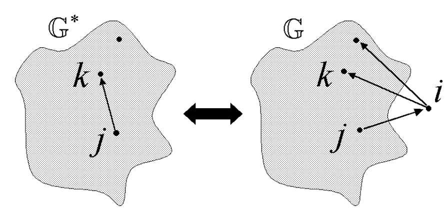

Fig. 14. Edge Splitting - directed case.

condition iii). The newly inserted vertex i is of out-degree 2. These edges do not change the out-degree of other vertices. In the replacement of the edge (j, k) by (j, i), the out-degree of j is also preserved. □

Example 3.8: The edge splitting operation for a directed graph is shown in Figure 14.

When edge splitting does not lead to a cycle, the resulting graph can always be created by using only vertex addition (from Proposition 3.3 and Corollary 3.4). Hence all stably rigid acyclic formations can be created by using the vertex addition operation.

IV. Cycles in Directed Formations

In undirected formations, both agents at the end-points of a sensing/communication link maintain a set-distance between each other. For this reason, an undirected link can be considered as two directed links between these two agents with opposite directions, and the underlying graph of an undirected formation can be considered to be a directed multigraph where each link is replaced by a cycle of length 2 between the end-points of the link. An example is shown in Figure 15. Let us assume that the desired distance between point agent i and point agent j is d_{ij}. The actual distance between these two point agents is $\|p_i - p_j\|$. In real applications, there are measurement errors, for instance due to noise, and it is reasonable to assume that these two agents will have different measurement errors. Let us assume that agent i has a constant measurement error of n_i and agent j has a constant measurement error of n_j.

Therefore the measured distance by agent i is $\|p_i - p_j\| + n_i$, and the measured distance by agent j is $\|p_i - p_j\| + n_j$. When agents reach to positions where they satisfy the distance constraint between each other, we would expect that agent i satisfies $\|p_i - p_j\| + n_i = d_{ij}$, and agent j satisfies $\|p_i - p_j\| + n_j = d_{ij}$. We assume that agents act autonomously in a decentralized, non-communicating way. If $n_i \neq n_j$, then there is no way that these two agents will reach positions such that the distance constraint is satisfied by both agents. The agents push and pull each other, or divert to infinity. Notice that this is a result of inconsistency created by noise in an overdetermined set of equations.

We say that a formation $(\mathcal{V}, \mathcal{L}, f)$ is realizable if there exists a mapping $\delta : \mathcal{V} \mapsto \mathbb{R}^{2n}$. If a formation is flexible, it has infinite number of realizations. If a formations is rigid, it has finite number of realizations. If a formation has a unique realization then it is called *globally rigid*. We refer the reader to [23] for an extensive treatment of globally rigid formations. If the underlying graph of a minimally rigid formation is rigid, it may still have no realization for a given set of link

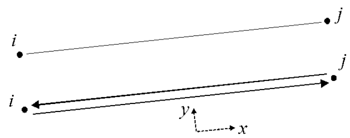

Fig. 15. A link in an undirected formation can be represented by two directed links. Thus it forms a 2-cycle in 1-dimensional space. This results in redundant rigidity in 1-dimensional space. Measurement errors cause redundancy-based inconsistencies in this overdetermined system.

lengths. This is due to a choice of impermissible link lengths, e.g., the triangle inequality is not satisfied. If a formation is redundantly rigid, a choice of impermissible link lengths, e.g., the triangle inequality, is still a source of the absence of a realization. However, in redundantly rigid formations, there is another source that may lead to the absence of a realization. This is due to the fact that, in a redundantly rigid formation, the set of equations for link lengths is overdetermined and almost any noise in measurements results in inconsistencies. Inconsistencies in overdetermined systems caused by redundant rigidity are called *redundancy-based inconsistencies*. Thus noisy measurements are a source of redundancy-based inconsistencies that disallow a realization in redundantly rigid formations. In the example given above with two links between two agents results in a redundantly rigid formation in 1-dimensional space (along the x-direction shown in Figure 15). One of the links is redundant, and noise in any links results in inconsistency. In 2-dimensional space, if an agent i has to satisfy three length constraints with agents a, b, and c, then one of the links is also redundant in terms of rigidity. If there is noise in the link length measurements, then there will be almost always no realization. This can be seen from the following set of constraint equations: $\|p_i - p_a\| = d_{ia}$, $\|p_i - p_b\| = d_{ib}$, $\|p_i - p_c\| = d_{ic}$. p_i is determined by the intersection of three circles centered at p_a, p_b, p_c with radii d_{ia}, d_{ib}, d_{ic}. If there is noise in link lengths, the set of equations for constraints are as follows: $\|p_i - p_a\| + n_{ia} = d_{ia}$, $\|p_i - p_b\| + n_{ib} = d_{ib}$, $\|p_i - p_c\| + n_{ic} = d_{ic}$. In that case, for almost all choices of n_{ia}, n_{ib}, n_{ic}, there is no solution for p_i, because three circles simply do not intersect at the same point. Thus we conclude that redundant rigidity cause inconsistencies if there are measurement errors. And the reason behind the inconsistency in a 2-cycle is redundancy in the set of constraints.

The behavior of agents on cycles of length 3 is strikingly different. Let us consider the formation shown in Figure 16. The underlying graph is minimally rigid. 1 is the global leader, 2 is the first-follower. Every other vertex has out-degree 2. It can be verified that there is a cycle of length 3 $(4, 5, 6, 4)$. Assume that the positions of agents 1, 2, 3 are fixed. Let us denote the desired distance between agent i and j by d_{ij}, and let us denote $p_i = (x_i, y_i)$. If there are no measurement errors, we expect the following hold:

$$\|p_i - p_j\| = d_{ij},$$

for $(i, j) \in \{(1,4),(2,5),(3,6),(4,5),(5,6),(6,4)\}$. Recall that p_1, p_2, p_3 are fixed. Therefore there are six unknowns in this set of equations. If the triangle inequality is satisfied, then we would expect that a solution exists. Consider the case where there are measurement errors denoted by n_{ij} on each link (i, j) on the cycle (4,6,5,4). We expect the following equations hold:

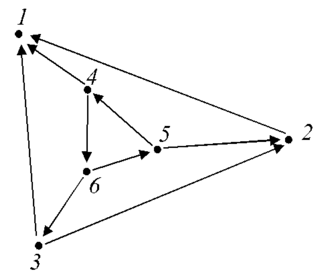

Fig. 16. A point formation that has a 3-cycle, $(4, 6, 5, 4)$. It is minimally rigid in 2-dimensional space, and has no redundancy contrary to the redundancy in the formation shown in Figure 15. Thus measurement errors do not cause redundancy-based inconsistencies in this formation.

$$\|p_i - p_j\| = d_{ij}, \text{ for } (i,j) \in \{(1,4),(2,5),(3,6)\}$$
$$\|p_i - p_j\| + n_{ij} = d_{ij}, \text{ for } (i,j) \in \{(4,5),(5,6),(6,4)\}$$

There is no reason to expect that introducing n_{ij}'s create an immediate inconsistency as it happened in the case of a 2-cycle. There are no redundant measurements so the system is not overdetermined. Note this is also true for formations that have cycles of length 4 and higher. If, for example, the triangle inequality is not satisfied, then the reason behind it is not cycles themselves, but rather selection of link lengths. Even acyclic formations with no noise can fail the triangle inequality if link lengths are poorly chosen.

One can also infer the existence of a solution under noisy measurements from the 'generic' property of rigidity and global rigidity in 2-dimensional space. When we perturb link lengths, the generic property of global rigidity ensures that there is still a realization for the formation.

A geometric interpretation is as follows: Consider the case with no noise in measurements in Figure 17(a). Given the positions p_1, p_2, and p_3 are fixed, then the points p_4, p_5, and p_6 are located such that all six equations are satisfied for the positions shown in Figure 17(a). Now, let us add noise to the edges $(4, 5)$, $(5, 6)$, and $(6, 4)$ as shown in Figure 17(b). Clearly, the current positions of point do not satisfy the measured link lengths. The new link lengths determine a unique triangle as shown in Figure 17(c). Can we locate this triangle such that its vertices touch the three circles but not cross the circles? The answer to this question is 'yes' provided the triangle inequality is satisfied. Therefore there is still a new set of solutions for the positions of points that satisfy the link lengths corrupted with noise. This is shown in Figure 17(d). For comparison purposes, the solution for the positions of points with no noise are denoted with empty circles in this Figure.

Figure 18 shows a formation with seven agents. It has three cycles, two of which have length 4, $(7, 6, 5, 3, 7)$ and $(4, 3, 7, 6, 4)$, and one of which has length 5, (4,3,7,6,5,4).

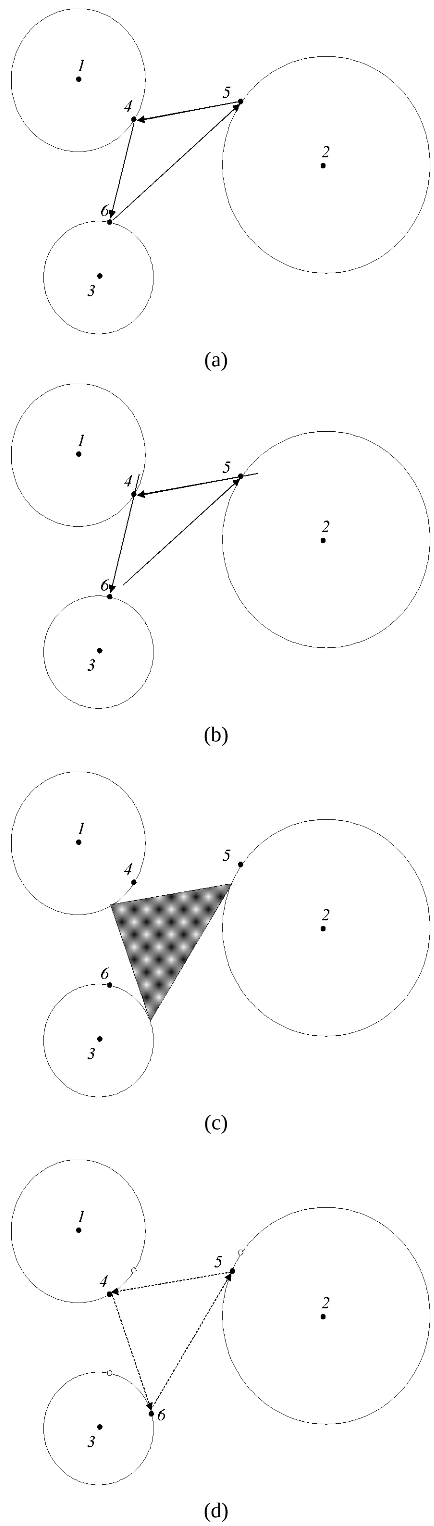

Fig. 17. (a) This figure shows the distance constraints that need to be satisfied by agents 4, 5, and 6 of the formation shown in Figure 16. The points in this figure clearly satisfy the constraints; (b) If noise is added to the lengths of links that lie on the 3-cycle $(4, 6, 5, 4)$, then points p_4, p_5, and p_6 fail to satisfy the distance constraints at their current positions; (c) The new distance constraints corrupted with noise can be represented by a triangle. It can be seen that the triangle can be placed between the circles in such a way that its vertices touch the circles at one single point, thus satisfying the distance constraints. (d) The vertices of this triangle determine the new locations of points p_4, p_5, and p_6 as shown with filled circles. The previous positions of the points are shown with empty circles.

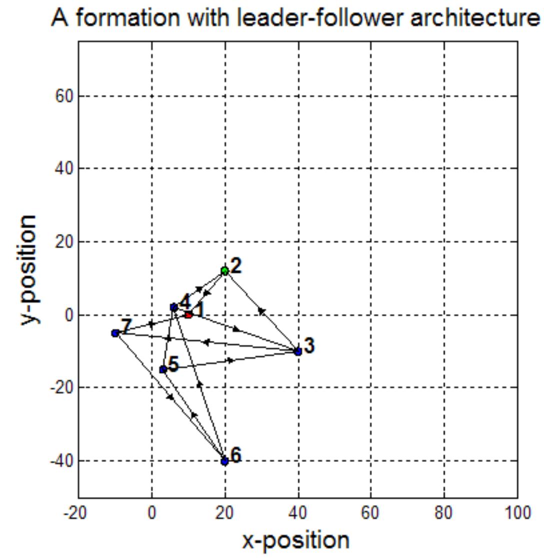

Fig. 18. A rigid formation created by vertex addition and edge splitting. The agent of out-degree 0 (global leader) is depicted with color red and has index 1. The agent of out-degree 1 is depicted with green and has index 2. The agents of out-degree 2 are depicted with color blue and has indices 3, 4, 5, 6, 7.

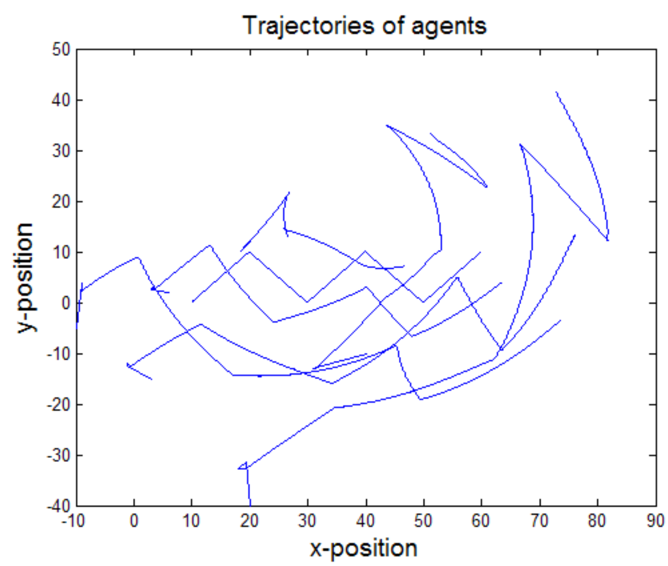

Fig. 19. Trajectories of agents in the formation shown in Figure 18.

The global leader moves on a zigzag trajectory. The plot of the trajectories of agents are shown in Figure 19. The distances between all agent pairs are shown in Figure 20.

Recall that there are three possibilities for a directed rigid formation: (i) a formation with all agents have out-degree 2 except a global leader of out-degree 0, a first follower of out-degree 1, and these two are connected by a link; (ii) a formation with all agents have out-degree 2 except a global leader of out-degree 0, a first follower of out-degree 1, and these two are not connected by a link; (iii) a formation with all agents of out-degree 2 except three agents of out-degree 1. As we stated before, case i) is the focus of this paper, because the positions of agents in case ii) and case iii) are not stable under

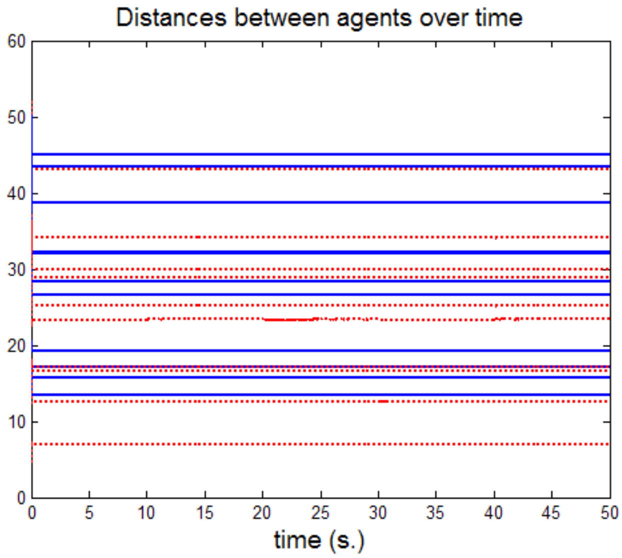

Fig. 20. Distances between agent pairs (of the formation shown in Figure 18) are shown over time as the global leader moves. The blue solid lines show the distances between agents where there exist links between those pairs. The red dotted lines show the distances between agent pairs where there are no links between those pairs.

small perturbations, i.e., translation and rotation of formations are easily affected by measurement and actuation errors. For directed formations that have cycles of length 3 or more, we have the following proposition. The structure of the overall proof will be clear with the algorithm given for selection of links in §V, therefore the proof is omitted here.

Proposition 4.1: A directed point formation in 2-dimensional space is stably rigid if and only if the following conditions hold: i) the underlying undirected graph is generically minimally rigid; ii) all vertices are of out-degree 2, except that exactly one vertex is of out-degree 0, exactly one vertex is of out-degree 1, and these two vertices are neighbors to each other.

V. Creating a Stably Rigid Directed Formation from a Rigid Undirected Formation

Apparently, stable rigidity of a directed formation depends not only on the underlying undirected formation but also on the directions of links between agents. Given a generically minimally rigid undirected formation, how do we find the directions of links to create a stably rigid directed formation? Below we present one way of doing this.

We start with giving preliminary definitions. A graph is *connected*, if there is a path from any vertex to any other vertex in the graph. A *tree* is a graph in which any two vertices are connected by exactly one path. A *spanning tree* of a connected, undirected graph is a tree which includes every vertex of that graph. There is a standard way of partitioning the edges in a generically minimally rigid graph with the following properties: (i) there are three trees; (ii) there are exactly two trees at each vertex; (iii) no two non-empty subtrees span the same set of vertices. These properties define a *3Tree2* partition of the edges [24], [25]. For a generically minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, it is also known that, for each $(i, j) \in \mathcal{L}$, the multigraph obtained by doubling the edge (i, j) is the union of two spanning trees [26].

Now we give a sequential algorithm to find the direction of links to create a stably minimally rigid directed formation from a minimally rigid undirected formation: (Let us assume that i represents the global leader, j represents the first follower connected to i by the edge (j, i).)

1) Double the edge (j, i) - The entire graph can now be partitioned into two spanning trees;
2) Remove (j, i) from one of the two trees - We now have 3-trees, one spanning, and one each containing the original two vertices;
3) Orient the spanning tree down to the selected leader;
4) Orient each of the other two trees down to the global leader or the first follower, whichever is in this revised tree.

This algorithm gives a stably rigid directed formation with out-degree 2 at each point except the first-follower of out-degree 1 and the global leader of out-degree 0. We give the following example to illustrate this algorithm:

Example 5.1: Consider the generically minimally rigid point formation shown in Figure 21(a). Assume that the global leader is labeled with 1 and the first follower is labeled with 2. The graph with the double edge $(2, 1)$ is shown in Figure 21(b). This graph can be partitioned into two spanning tress as shown in Figures 21(c) and 21(d). When we remove $(2, 1)$ from one of the two trees, in this case from Figure 21(d), we now have three trees: one spanning as shown in Figure 21(c), and one each containing the original two vertices as shown in Figures 22(a) and 22(b). Figure 23(a) shows the oriented the spanning tree down to the global leader. Figures 23(b) and 23(c) show the oriented two trees down to the global leader or the first follower. Finally, if we put together the edge topologies in Figures 23(a), 23(b), and 23(c), we obtain the directed point formation shown in Figure 23(d).

We note that this algorithm permits an arbitrary choice of the first edge in the graph. There is a way to deduce this directly from the assumption that the rigidity matrix has independent rows and full rank. This is implicit in the paper by White and Whiteley [22]. Effectively, if we take out the four columns for the vertices that correspond to the global leader and the first follower, and the row for their connecting edge, we have a square matrix. If this is minimally rigid, we can check this by taking the determinant - which must be non-zero. If we decompose this determinant into blocks, using 2×2 minors for the columns of each of the remaining vertices, as a Laplace Expansion, then we have a sum of products of such minors. Since the sum is non-zero, some term is non-zero. Each of these terms in the sum identifies two edges used in the block for a vertex. We identify those edges as ‘out’ for that vertex. This identifies a direction for the edges, with out-degree 2 for all but the vertices corresponding to the global leader and the first follower.

If the graph has a simple peeling down to the initial edge, then there will be only one term in this ‘sum’ - and the directed graph to the initial leading edge will be unique. For example, consider the graph shown in Figure 24 with the vertices $\{1, 2\}$ that represent the global leader and the first follower. It has

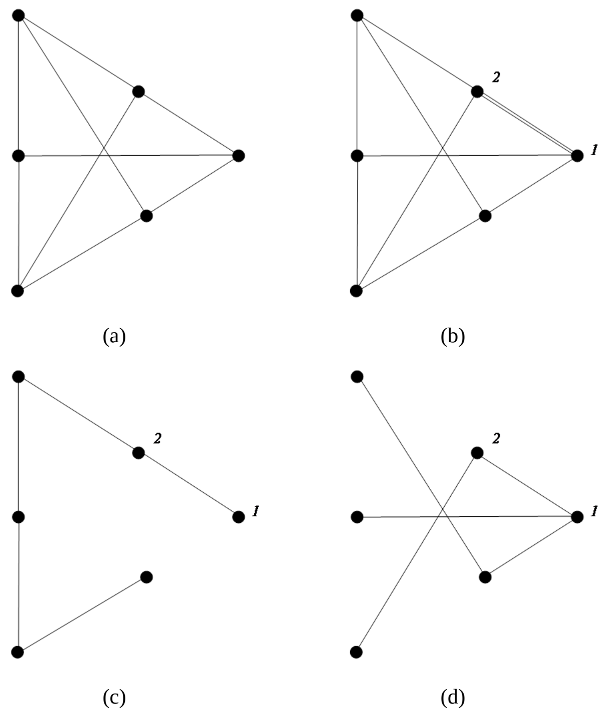

Fig. 21. A minimally rigid point formation is shown in (a). The graph with the double edge $(2, 1)$ is shown in (b). The global leader is labeled with 1 and the first follower is labeled with 2. The graph in (b) can be partitioned into two spanning trees as shown in (c) and (d).

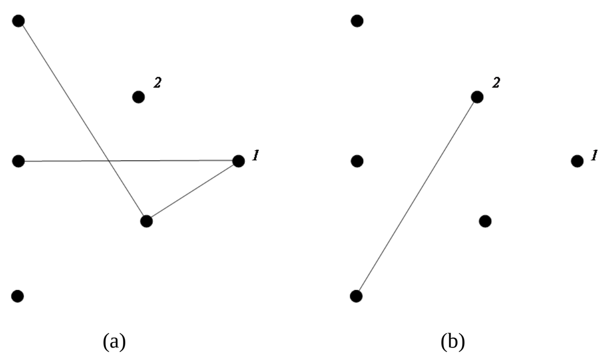

Fig. 22. When we remove $(2, 1)$ from one of the two spanning trees in Figures 21(c) and 21(d), in this case from Figure 21(d), we now have three trees: one spanning as shown in Figure 21(c), and one each containing the original two vertices as shown in (a) and (b) in this figure.

the simple peeling down to initial edge and it is shown in Figure 25. If the sum has more than one term, then the terms actually differ by re-orienting cycles in the one (and all) of the possible directed graphs with out-degree 2.

Given a generically minimally rigid graph $\mathbb{G}$, there may be no choice for leader-follower which generates a simple peeling. Given a graph, there may be some choices for the leader-follower edge which produce a simple peeling, and other choices which do not permit a simple peeling. For example the graph on the top right of Figure 26 shows a choice of leader-follower pair, which does not permit a simple peeling. If there is a simple peeling, for selected edge, then there is a unique orientation with out-degree at most 2 towards this edge.

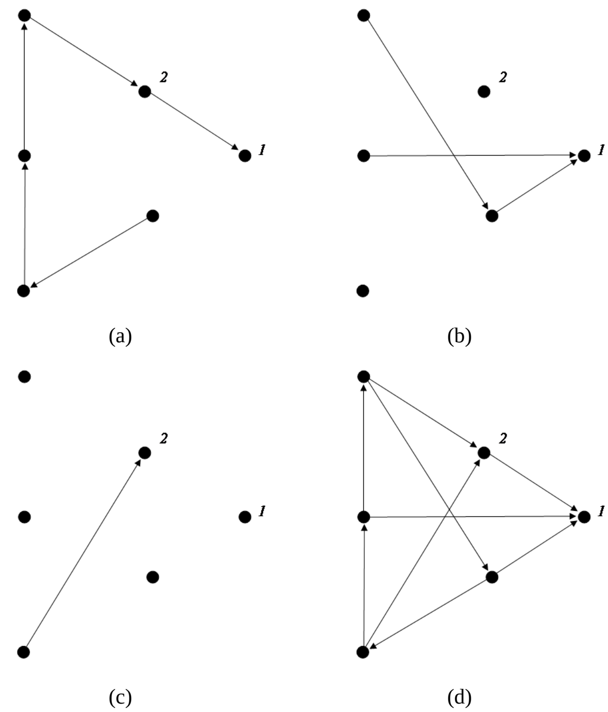

Fig. 23. The oriented spanning tree down to the global leader is shown in (a). The oriented two trees down to the global leader or the first follower are shown in (b) and (c). If we put together the edge topologies in (a), (b) and (c), we obtain the directed point formation shown in (d).

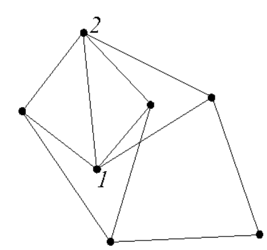

Fig. 24. A minimally rigid undirected point formation is shown. We pick the global leader and the first follower labeled with 1 and 2, respectively. We would like to know how to determine the directions of links once the undirected formation and the global leader-first follower pair are given.

If there is a directed graph towards a selected follower-leader edge, with out-degree 2 on all other vertices, which has a cycle, then all directed graphs towards this follower-leader edge also have a cycle. In fact, any two such graphs are interchanged by reorienting a finite set of simple cycles in one of the graphs.

What about cycles vs. no cycles? No cycles and minimal rigidity, requires the construction sequence using only vertex addition - a ‘simple peeling down’ if reversed (see above). Once we have such a sequence, then all directed graphs with out-degree at least 2 for all but the ‘leading vertices’ will also have no cycles, or equivalently, with no vertex of out-degree greater than 2, will also have no cycles, and all acyclic coverings will have out-degree at most 2.

Moreover, this follows for any other choice of the initial edge. All choices of an initial edge result in a simple peeling down, and a unique acyclic directed graph to this edge

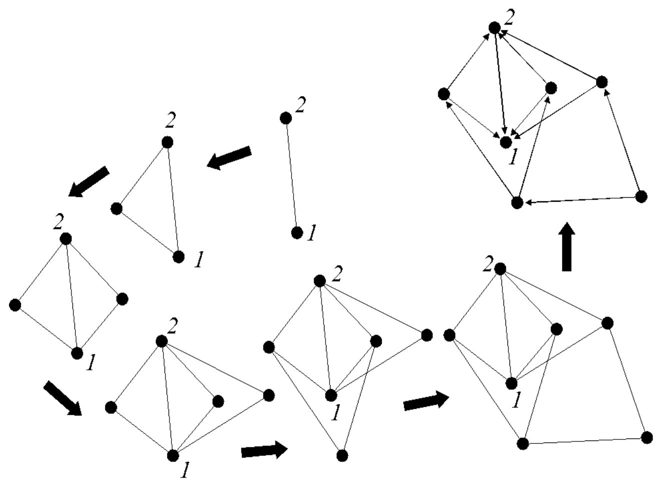

Fig. 25. Once the undirected graph, the global leader-first follower pair are given, this figure shows a simple peeling down sequence to determine the directions of links.

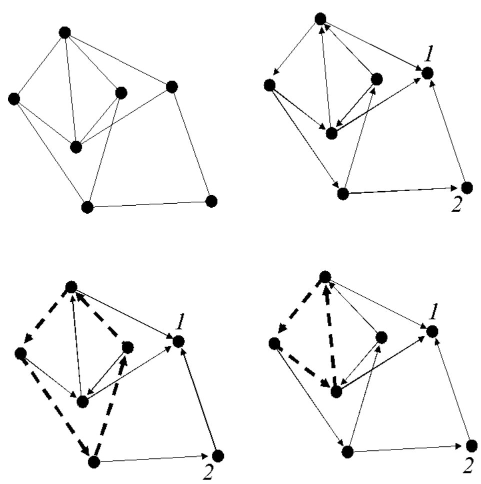

Fig. 26. Given the point formation shown top left, if the global leader labeled with 1 and the first follower labeled with 2 are chosen as shown on top right, then we obtain a cyclic rigid formation. Cycles are shown on bottom right and bottom left with dotted edges. This selection of global leader-first follower pair results in only cyclic formations, and does not permit any acyclic formations.

(equivalently, a unique directed graph with out-degree at most 2 down to this edge).

The converse is that if there is one directed graph with out degree at most 2 that does have a cycle, then there is no simple peeling down for any choice of initial edge. Therefore, there is no other orientation with out-degree at most 2 that has no cycles, or no acyclic orientation which has out-degree at most 2.

Given a generically minimally rigid graph $\mathbb{G}$, there is a fast (the worst case order $|\mathcal{V}||\mathcal{L}|$) algorithm (the pebble game) which can generate:

- an orientation towards a selected leader-follower edge with out-degree 2 on all other vertices;
- can switch from one such choice to any graph for another leader-follower edge in linear time;
- can detect whether there is an acyclic orientation for this edge.

Given a set of vertices and independent edges, the algorithm can select additional edges to extend this to an (oriented) minimally rigid graph, in order $|\mathcal{V}|$ time. Given a rigid graph, the algorithms can select a minimally rigid sub-graph in order $|\mathcal{L}|$ time.

VI. Concluding Remarks

Propositions 3.3 and 4.1 provide the necessary and sufficient conditions for stable rigidity in directed formations. The algorithm given in §V establishes a sequential way of determining the directions of links from a given undirected rigid formation so that the necessary and sufficient conditions are fulfilled. We expect that the framework given in this paper will be useful in analysis of formation rigidity and stability problems and will be a useful tool to create stably rigid directed formations.

We want to draw the attention of the reader that this paper suggests both cyclic and acyclic directed formations can be stably rigid. The question of whether one is better than the other or both types are equivalent in terms of the performance of stable rigidity is still an open question. Qualitatively we can say that acyclic formations seem much simpler to work with compared to cyclic formations, but this question requires more quantitative analysis. From a topological point of view, acyclic formations are easier to work with for more decentralized and local operations, such as agent departures, formation splitting and merging. On the contrary, cyclic formations can easily get complicated for such operations. Furthermore, the set point of an agent on a cycle depends its own position. A quantitative analysis of convergence of agent positions to set points will be useful to realistically compare cyclic and acyclic formations.

Although cyclic formations are more difficult to work with at the topological level, they have some advantages over acyclic formations. Acyclicity provides position information flow only in one direction, thus reduces the level of cooperation effort between agents. If an agent located at the back of an acyclic formation fails, there is no way that leaders can realize this failure. On the other hand, cycles can provide feedback among the leaders and followers in a formation, thus increasing the coherence between formation members.

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