Converting from Spherical to Parabolic Coordinates

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We start with θ and φ, as defined in computer graphics so that θ is the angle from the pole, or latitude, and φ is the polar angle, or longitude. This is the opposite of mathematical convention.

Parabolic coordinates are defined for a hemisphere as normalized Cartesian vector, <x, y, z>, as:

$$u = \frac{x}{z+1} \quad (1.1)$$

$$v = \frac{y}{z+1} \quad (1.2)$$

Given a direction described as <θ, φ>, the normalized Cartesian vector is given as:

$$x = \cos\phi \sin\theta \quad (2.1)$$

$$y = \sin\phi \sin\theta \quad (2.2)$$

$$z = \cos\theta \quad (2.3)$$

Combining equations (1.x) with (2.x) we get:

$$u = \frac{\cos\phi \sin\theta}{\cos\theta + 1} \quad (3.1)$$

$$v = \frac{\sin\phi \sin\theta}{\cos\theta + 1} \quad (3.2)$$

For the reduction to a simpler form, we use the following trigonometric identities:

$$\frac{\sin t}{\cos t} = \tan t \quad (4.1)$$

$$\frac{1+\cos(2t)}{2} = \cos^2 t \quad (4.2)$$

$$\sin(2t) = 2\sin t \cos t \quad (4.3)$$

We define $\theta^{'}=(\theta/2)$, and rewrite equations (3.x) as:

$$u = \frac{\cos\phi \sin\left(2\theta^{'}\right)}{\cos\left(2\theta^{'}\right)+1} \quad (5.1)$$

$$v = \frac{\sin\phi \sin\left(2\theta^{'}\right)}{\cos\left(2\theta^{'}\right)+1} \quad (5.2)$$

Using equation (4.2), in equations (5.x), we get:

$$u = \frac{\frac{1}{2}\cos\phi \sin\left(2\theta^{'}\right)}{\cos^2\theta^{'}} \quad (6.1)$$

$$v = \frac{\frac{1}{2}\sin\phi \sin\left(2\theta^{'}\right)}{\cos^2\theta^{'}} \quad (6.2)$$

Substituting (4.3) into (6.x), we further get:

$$u = \frac{\frac{1}{2}\cos\phi 2\sin\theta^{'}\cos\theta^{'}}{\cos^2\theta^{'}} \quad (7.1)$$

$$v = \frac{\frac{1}{2}\sin\phi 2\sin\theta^{'}\cos\theta^{'}}{\cos^2\theta^{'}} \quad (7.2)$$

Some algebraic simplification yields:

$$u = \frac{\cos\phi \sin\theta^{'}}{\cos\theta^{'}} \quad (8.1)$$

$$v = \frac{\sin\phi \sin\theta^{'}}{\cos\theta^{'}} \quad (8.2)$$

A final substitution of (4.1) into (8.x), and remembering the definition of $\theta^{'}$, gives us:

$$u = \cos\phi \tan\left(\theta/2\right) \quad (9.1)$$

$$v = \sin\phi \tan\left(\theta/2\right) \quad (9.2)$$

As a final verification, we know that *u* and *v* range from [-1,1]. Looking at 9.x, we see that tan(θ/2) will be in the range [0,1] for θ in [0,π].

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