What is the complexity of volume calculation?

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Abstract

We study the worst case complexity of computing ε-approximations of volumes of d-dimensional regions $g([0,1]^d)$, by sampling the function g. Here, g is an s times continuously differentiable injection from $[0,1]^d$ to $\mathbb{R}^d$, where we assume that $s \geq 1$. Since the problem can be solved exactly when $d = 1$, we concentrate our attention on the case $d \geq 2$. This problem is a special case of the surface integration problem studied in [12]. Let $\mathbf{c}$ be the cost of one function evaluation. The results of [12] might suggest that the ε-complexity of volume calculation should be proportional to $\mathbf{c}(1/\varepsilon)^{d/s}$ when $s \geq 2$. However, using integration by parts to reduce the dimension, we show that if $s \geq 2$, then the complexity is proportional to $\mathbf{c}(1/\varepsilon)^{(d-1)/s}$. Next, we consider the case $s = 1$, which is the minimal smoothness for which our volume problem is well-defined. We show that when $s = 1$, an ε-approximation

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can be computed with cost proportional to at most $\mathbf{c}(1/\varepsilon)^{(d-1)d/2}$. Since a lower bound proportional to $\mathbf{c}(1/\varepsilon)^{d-1}$ holds when $s = 1$, it follows that the complexity in the minimal smoothness case is proportional to $\mathbf{c}(1/\varepsilon)$ when $d = 2$, and that there is a gap between the lower and upper bounds when $d \geq 3$.

1 Introduction

The approximation of volumes is an important computational problem. There are several different approaches in the literature. One approach is to assume that we have *complete* information about the region. For example, [1] discusses the complexity of computing volumes of d-dimensional closed, orientable polyhedra P in the worst case setting using the real number model. If ∂P has been triangulated into a set $\mathscr{T}$ of $(d-1)$-simplices, then the volume of P can be calculated with cost roughly proportional to $\frac{1}{3}d|\mathscr{T}|$. Noting that $|\mathscr{T}| \gg d$ often holds, they show that the cost is proportional to $2.6d^2$, assuming that ∂P can be traversed by moving between $(d-1)$-simplices sharing a common $(d-2)$-face.

Exact volume calculation has also been studied in the Turing model of computation. Khachiyan [8] proved that calculating the volumes of polytopes is NP-hard. For more powerful negative results in this directions, see the references in [7].

Another area of active research is the approximation of volumes of convex sets in the real number model, see [7] for a review. In this case, we use only *partial* information, often given by membership tests. That is, we can check whether a given point belongs to a convex set. Sometimes, this information is strengthened by requiring the knowledge of a separating hyperplane when the point does not belong to the convex set. Usually, randomized algorithms are considered. One result along these lines is given by [7]. They show that the volume can be approximated with relative error at most ε with probability at least $1-\eta$, with cost $O(d^5/\varepsilon^2(\ln 1/\varepsilon)^3(\ln 1/\eta)\ln^5 d)$.

Another approach to approximating the volume of a region is to replace it by a simpler region. A typical technique is to use a piecewise polynomial approximation of a region's boundary, and then to use exact formulas to calculate the volume of the approximating region. This approach also uses the real number model. See [2] for an example, as well as for references to the relevant literature.

We now explain our approach to this problem. We study the worst case complexity of calculating volumes of regions in the real number model. Here, our regions are of the form $g(I^d)$, where g belongs to a given class of functions defined over $I^d = [0,1]^d$. Only partial information, given by finitely many values of g, is available. This kind of information is different from membership tests, since it only delivers points belonging to the region. It is also more general than boundary information, since it can include points inside the region.

In this paper, we will consider classes G_s of s times continuously differentiable injections of I^d. Hence, we are approximating volumes of regions $g(I^d)$, where $g \in G_s$. Such regions are diffeomorphic images of the unit cube I^d. Since the cube I^d has corners, this means that the region $g(I^d)$ will not have a globally smooth boundary. In particular, this means that the d-dimensional Euclidean unit ball B_d does *not* belong to G_s. The complexity of calculating volumes of diffeomorphic images of B_d is an open problem, which we hope to handle in the future.

The volume of $g(I^d)$ is equal to the integral of the Jacobian determinant of g over I^d. We want to use this characterization as our point of departure. Since the Jacobian determinant is well-defined iff $s \geq 1$, we shall restrict our attention in this paper to the case $s \geq 1$.

For the univariate case $d = 1$, the volume problem is trivial, and can be solved exactly using two evaluations of g; moreover, we prove that two evaluations are necessary.

Therefore, we concentrate our attention on the case $d \geq 2$. This problem is a special case of the surface integration studied in [12]. This paper supplies upper bounds for the volume problem only for the case $s \geq 2$. These upper bounds are of the order $\mathbf{c}(1/\varepsilon)^{d/s}$, where $\mathbf{c}$ is the cost of one function evaluation. Our initial expectation was that these bounds would be sharp; however, our intuition was wrong. Since the Jacobian determinant can be expressed in a divergence form, see [5, Chapter 4, Theorem 3.2], we can use integration by parts to reduce the dimension. This yields an upper bound proportional to $\mathbf{c}(1/\varepsilon)^{(d-1)/s}$, still assuming that $s \geq 2$. Is this upper bound sharp? Indeed, it is. We show that for any $s \geq 1$, the volume problem is no easier than the $(d-1)$-dimesional problem of integrating s times continuously differentiable functions defined over I^{d-1}. The latter problem is known to have complexity of order $\mathbf{c}(1/\varepsilon)^{(d-1)/s}$, see [3] as well as [9] and [11]. Hence, the complexity of the volume problem is also of order $\mathbf{c}(1/\varepsilon)^{(d-1)/s}$ when $s \geq 2$.

Let us now consider the remaining case $s = 1$, that is, the functions g determining our regions are only continuously differentiable. The Jacobian of g is merely continuous. Since the complexity of integrating continuous functions is infinite, see [3], [9] and [11], it was unclear whether this volume problem could be solved with finite complexity.

We have only partial results for this case $s = 1$. The good news is that the complexity is finite for any d, and is at most of the order $\mathbf{c}(1/\varepsilon)^{(d-1)d/2}$. The bad news is that we know that this upper bound is sharp only for the case $d = 2$, for which we see that the complexity is of the order $\mathbf{c}(1/\varepsilon)$. When $d \geq 3$, there is a gap between the lower and upper bounds, which we have been unable to bridge.

We briefly review the contents of this paper. In Section 2, we present the formal definition of the volume problem. In Section 3, we present the easy univariate case. Section 4 is the major part of this paper, dealing with the multivariate case. We first present a lower bound for the case $s \geq 1$. Next, we present an upper bound for the case $s \geq 2$, using an algorithm based on the surface integration algorithm of [12]. The final subsection deals with the case $s = 1$. We present and analyze an algorithm for this case. This algorithm is substantially different than that for the smoother case $s \geq 2$, and is defined by induction on d.

2 Problem formulation

Before describing the problem to be solved, we first recall the definition of the volume of a region; see [6, pg. 334 ff.] for further discussion. We let $I = [0, 1]$ denote the unit interval, so that $I^d = [0, 1]^d$. Let d be a given positive integer. For a C^1 injection $g\colon I^d \to \mathbb{R}^d$, the set

$$g(I^d) = \{ g(x) : x \in I^d \}$$

is a d-dimensional *region* whose volume we want to approximate by sampling the function g. Using the standard change of variables formula, the *volume of* $g(I^d)$ is

$$\operatorname{vol} g(I^d) = \int_{I^d} |(\det \nabla g)(x)| \, dx, \tag{2.1}$$

with the *gradient* $\nabla g\colon I^d \to \mathbb{R}^{d\times d}$ being defined as[1]

$$[(\nabla g)(x)]_{i,j} = (\partial_j g_i)(x) \qquad \text{for } i, j \in \{1, \ldots, d\} \text{ and } x \in I^d.$$

We now describe the problem to be solved. Let G be a class of C^1 injections having domain I^d and codomain $\mathbb{R}^d$. We want to approximate the *volume* operator defined by

$$S(g) = \operatorname{vol} g(I^d) \qquad \forall\, g \in G.$$

Note that S is a nonlinear functional.

We compute an approximation $U(g)$ to $S(g)$ by using information

$$N(g) = \left[g_{i_1}(x^{(1)}), \ldots, g_{i_n}(x^{(n)})\right] \tag{2.2}$$

where

$$x^{(j)} = \left(x_1^{(j)}, \ldots, x_d^{(j)}\right) \qquad (1 \leq j \leq n)$$

[1] Here, ∂_j denotes the partial derivative in the jth coordinate direction and g_i is the ith component of g.

and

$$i_1, \dots, i_n \in \{1, \dots, d\}.$$

We also allow adaption. That is, the number $n = n(g)$ of evaluations, as well as the sample points $x^{(1)}, \dots, x^{(n)}$, may depend on the previously-computed function values of g; for details, see, e.g., [10, Chapter 2]. We let

$$\operatorname{card} N = \sup_{g \in G} n(g)$$

denote the *cardinality* of the information N.

Remark. Note that the permissible information is given by evaluating $g_1, \dots, g_d$ at points in I^d. One could also allow the evaluation of partial derivatives of the g_i, as well. We restrict ourselves to function values alone, as this makes the exposition much simpler. However, it is easy to see that the results of this paper also hold if arbitrary partial derivative evaluations are allowed. □

Our approximation U is given by

$$U(g) = \phi\big(N(g)\big) \tag{2.3}$$

for some mapping $\phi\colon N(G) \to \mathbb{R}$. The *worst case error* of an approximation is defined to be

$$e(U) = \sup_{g \in G} |S(g) - U(g)|.$$

The cost of computing $U(g)$ is defined as $\operatorname{cost} U(g)$, which is the weighted sum of the total number of function values of $g_1, \dots, d_d$, as well as the number of arithmetic operations and comparisons needed to obtain $U(g)$. More precisely, we assume that for any $i \in \{1, \dots, d\}$, the evaluation of g_i costs $\mathbf{c}$. The cost of each arithmetic operation is taken as 1. For U of the form (2.3), we have

$$\operatorname{cost} U(g) = \mathbf{c}\, n + \tilde{n},$$

where $\tilde{n}$ is the total number of arithmetic operations and comparisons needed to compute $U(g)$, given $N(g)$. Here $\mathbf{c} \ge 1$, and usually it is realistic to assume that $\mathbf{c} \gg 1$; see once more [10, Chapter 2] or [11, Chapter 2] for details. Then

$$\operatorname{cost} U = \sup_{g \in G} \operatorname{cost} U(g)$$

is the *worst case cost* of U.

We may judge the quality of an approximation U using information of given cardinality by comparing its error to the minimal error possible among all approximations using information of the same cardinality. For fixed n, the *nth minimal error*

$$e(n) = \inf\{\, e(U) : U \text{ of the form (2.3) with } \operatorname{card} N \le n \,\}$$

is the minimal error among all approximations using any information of cardinality at most n. Clearly, $\{e(n)\}$ is a nonincreasing sequence. Moreover, $e(n)$ makes sense even when $n = 0$; indeed, $e(0)$ is minimal error among all "constant" approximations, i.e., those using no evaluations of g.

Along with minimal-error approximations using a given number n of information evaluations, we also wish to obtain ε-approximations at minimal cost for any $\varepsilon \geq 0$. The ε*-complexity* of volume computation is the minimal cost of computing an ε-approximation, i.e.,

$$\mathrm{comp}(\varepsilon) = \inf\{\, \mathrm{cost}\, U \;:\; U \text{ such that } e(U) \leq \varepsilon \,\}.$$

An approximation U_ε for which[2]

$$e(U_\varepsilon) \leq \varepsilon \qquad \text{and} \qquad \mathrm{cost}\, U_\varepsilon \asymp \mathrm{comp}(\varepsilon) \quad \text{as} \quad \varepsilon \to 0,$$

is said to be (asymptotically) *optimal*.

Remark. The error $e(U)$, the nth minimal error $e(n)$, and the ε-complexity also depend on the class G of problem elements. Where necessary, we shall show this dependence explicitly, by writing, e.g., $e(n; G)$ for $e(n)$ and $\mathrm{comp}(\varepsilon; G)$ for $\mathrm{comp}(\varepsilon)$. □

The purpose of this paper is to find sharp estimates of the nth minimal error and the ε-complexity of volume calculation, as well as optimal algorithms.

We will chose a specific class $G_s = G_{s;d,m,M}$ as our class G of problem elements. This class will consist of all functions $g\colon I^d \to \mathbb{R}^d$ that are s times continuously differentiable and satisfy

$$\|g\|_{C^s(I^d;\mathbb{R}^d)} \leq M$$

and

$$\inf_{x\in I^d} |(\det \nabla g)(x)| > m.$$

Here s is a fixed positive integer and, for any positive integer l, the norm $\|\cdot\|_{C^s(I^d;\mathbb{R}^l)}$ is given by[3]

$$\|g\|_{C^s(I^d;\mathbb{R}^l)} = \max_{|\alpha|\leq s} \max_{1\leq i\leq l} \|D^\alpha g_i\|,$$

with $\|\cdot\|$ in the right-hand side of the line above denoting the max norm. Moreover, the parameters m and M satisfy

$$0 \leq m < 1 \leq M.$$

[2]We use $\preccurlyeq$, $\succcurlyeq$, and $\asymp$ in this paper to respectively denote O-, Ω-, and Θ-relations.

[3]We use the standard notation for multi-indices and for Sobolev spaces, norms, and seminorms, see (e.g.) [4]. In particular, for an integer multi-index $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_d]$, we have $D^\alpha = \partial^{|\alpha|}/(\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d)$.

In particular, note that the identity mapping $x \mapsto x$ belongs to G_s.

Our estimates will be sharp only in terms of the power of ε^{-1}, with constants depending on s, d, m, and M.

In what follows, it will be useful to introduce an auxiliary solution operator $S_d \colon G \to \mathbb{R}$, defined as

$$S_d(g) = \int_{I^d} (\det \nabla g)(x)\, dx.$$

Thus, $S_d(g)$ is the *signed volume* of $g(I^d)$. Concepts such as nth minimal error and ε-complexity for S are defined just as they were for the volume problem; where necessary, we shall indicate this notationally. Note that we pay special attention to how the signed volume depends on dimension; this is because we will be using a reduction of dimension to attain our approximations to S_d.

It is easy to see that by the definition of the class G_s, we have

$$S(g) = |S_d(g)| \qquad \forall\, g \in G_s.$$

That is,

$$S_d(g) = \pm S(g) \qquad \forall\, g \in G_s.$$

with the choice of using a plus or a minus sign being determined with one evaluation of $(\det \nabla g)(x)$. It is easy to check that

$$e(n; S, G_s) = e(n; S_d, G_s) \qquad \forall\, n \geq 0$$

and that

$$\mathrm{comp}(\varepsilon; S, G_s) = \mathrm{comp}(\varepsilon; S_d, G_s) + O(1) \qquad \text{as } \varepsilon \to 0.$$

Thus the complexity of the volume and signed volume problems is essentially the same.

3 The case $d = 1$

We now show that the univariate case $d = 1$ is trivial. It can be solved exactly using two evaluations of g. However, if fewer than two evaluations are used, the problem cannot be solved exactly.

Theorem 3.1. *Let $d = 1$.*

1. *The nth minimal error $e(n; S, G_s)$ is positive for $n = 0$ and $n = 1$.*

2. *For $n \geq 2$, the nth minimal error $e(n; S, G_s)$ is zero, and the approximation U given by*

$$U(g) = |g(1) - g(0)| \qquad \forall\, g \in G_s \tag{3.1}$$

is optimal and has zero error.

Proof. First, consider the cases $n = 0$ and $n = 1$. Since $e(0) \geq e(1)$, it suffices to show that

$$e(1; S, G_s) \geq \tfrac{1}{2}(M - m) > 0. \tag{3.2}$$

To this end, let N be information of cardinality at most one. Define $g(x) = Mx$ and write $N(g) = [g(x^{(1)})]$. Choose $\overline{m} \in (m, 1)$. Let $\tilde{g}(x) = \overline{m}x + (M - \overline{m})x^{(1)}$. Then $g, \tilde{g} \in G_s$, with $N(\tilde{g}) = N(g)$. Then

$$S(g) - S(\tilde{g}) = M - \overline{m}.$$

From [10, pg. 45], we know that

$$e(1; S) \geq \inf_{x^{(1)} \in I^d} \tfrac{1}{2}|S(g) - S(\tilde{g})| = \tfrac{1}{2}(M - \overline{m}).$$

Since $\overline{m}$ may be chosen arbitrarily close to m, we see that (3.2) holds, as claimed.

To prove the result for $n \geq 2$, it suffices to show that the approximation (3.1) has zero error. To see this, let $g \in G_s$. Then g' is a continuous function that never vanishes, and so either $g' > 0$ in I, which holds if $g(1) > g(0)$, or $g' < 0$ in I, which holds if $g(1) < g(0)$. In the former case, we have

$$S(g) = \int_0^1 |g'(x)|\, dx = \int_0^1 g'(x)\, dx = g(1) - g(0),$$

and in the latter case, we have

$$S(g) = \int_0^1 |g'(x)|\, dx = \int_0^1 -g'(x)\, dx = g(0) - g(1).$$

Hence in either case, we have

$$S(g) = |g(1) - g(0)| = U(g),$$

and so the approximation given by (3.1) is optimal and has zero error. □

4 The case $d \geq 2$

In this section, we consider the multivariate case $d \geq 2$. We first establish a lower bound.

Theorem 4.1. *For $d \geq 2$, the nth minimal error satisfies*

$$e(n; S) \succcurlyeq \left(\frac{1}{n}\right)^{s/(d-1)}.$$

Proof. Let N be information of cardinality at most n. Choose $\overline{m} \in (m, 1)$. Define the function $g \in C^s(I^d; \mathbb{R}^d)$ as

$$g(x) = [x_1, \dots, x_{d-1}, \tfrac{1}{2}(1 + \overline{m})x_d] \qquad \forall\, x \in I^d.$$

Since $m < 1 \leq M$, we find

$$\|g\|_{C^s(I^d;\mathbb{R}^d)} \leq M$$

and

$$(\det \nabla g)(x) = \tfrac{1}{2}(1 + \overline{m}) > m.$$

Hence $g \in G_s$, and

$$S(g) = \tfrac{1}{2}(1 + \overline{m}).$$

As in (2.2), we write

$$N(g) = [g_{i_1}(x^{(1)}), \dots, g_{i_\ell}(x^{(\ell)})],$$

where $\ell \leq n$. Note that ℓ, as well as the selection of the points $x^{(1)}, \dots, x^{(\ell)}$, may be determined adaptively. Let us write

$$y^{(j)} = (x_1^{(j)}, \dots, x_{d-1}^{(j)}) \qquad \text{for } 1 \leq j \leq \ell.$$

From [3], see also [9], we can find a function $w \colon I^{d-1} \to \mathbb{R}$ satisfying

$$\begin{gathered} w(y^{(1)}) = \dots = w(y^{(\ell)}) = 0, \\ \|w\|_{C^s(I^{d-1};\mathbb{R})} = 1, \\ \int_{I^{d-1}} w(x_1, \dots, x_{d-1})\, dx_1 \, \dots \, dx_{d-1} \succcurlyeq \left(\frac{1}{\ell}\right)^{s/(d-1)} \geq \left(\frac{1}{n}\right)^{s/(d-1)}. \end{gathered} \tag{4.1}$$

Let

$$z(x) = x_d\, w(x_1, \dots, x_{d-1}).$$

Since $\|z\|_{C^s(I^d;\mathbb{R})} = \|w\|_{C^s(I^{d-1};\mathbb{R})}$, we have

$$\|z\|_{C^s(I^d;\mathbb{R})} = 1$$

and

$$z(x^{(1)}) = \cdots = z(x^{(\ell)}) = 0.$$

Let

$$\tilde{g}(x) = [x_1, \ldots, x_{d-1}, \tfrac{1}{2}(1+\overline{m})x_d + \tfrac{1}{2}(1-\overline{m})z(x)] \qquad \forall\, x \in I^d.$$

Then

$$\|\tilde{g}\|_{C^s(I^d;\mathbb{R}^d)} \le 1 < M,$$

and

$$\begin{aligned}(\det \nabla\tilde{g})(x) &= \tfrac{1}{2}(1+\overline{m}) + \tfrac{1}{2}(1-\overline{m})(\partial_d z)(x)\\ &\ge \tfrac{1}{2}(1+\overline{m}) - \tfrac{1}{2}(1-\overline{m})\|z\|_{C^s(I^d;\mathbb{R})}\\ &= \tfrac{1}{2}(1+\overline{m}) - \tfrac{1}{2}(1-\overline{m}) = \overline{m} > m.\end{aligned}$$

Hence $\tilde{g} \in G_s$, and $N(\tilde{g}) = N(g)$. Once again using [10, pg. 45], along with (4.1), we see that

$$\begin{aligned}2\,e(N) \ge S(\tilde{g}) - S(g) &= \tfrac{1}{2}(1-\overline{m}) \int_{I^d} (\partial_d z)(x)\,dx\\ &= \tfrac{1}{2}(1-\overline{m}) \int_{I^{d-1}} [z(x_1,\ldots,x_{d-1},1) - z(x_1,\ldots,x_{d-1},0)]\,dx_1\,\ldots\,dx_{d-1}\\ &= \tfrac{1}{2}(1-\overline{m}) \int_{I^{d-1}} w(x_1,\ldots,x_{d-1})\,dx_1\,\ldots\,dx_{d-1} \succcurlyeq \left(\frac{1}{n}\right)^{s/(d-1)}.\end{aligned}$$

Since N is arbitrary information of cardinality at most n, the desired result now follows. □

We now turn to establishing upper bounds. Before doing this, we establish a more convenient form for the solution operator S_d. To do this, we also need another auxiliary operator $\tilde{S}_d$, defined as

$$\tilde{S}_d(f, w) = \int_{I^d} f(x)(\det \nabla w)(x)\,dx \tag{4.2}$$

for $f \in C^1(I^d;\mathbb{R})$ and $w \in C^1(I^d;\mathbb{R}^d)$. For $g \in C^1(I^d;\mathbb{R}^d)$, $j \in \{1,\ldots,d\}$, and $a \in [0,1]$, define the mappings $g_{1,j,a}\colon I^{d-1} \to \mathbb{R}$ and $g_{j,a}\colon I^{d-1} \to \mathbb{R}^{d-1}$ as

$$g_{1,j,a}(x) = g_1\left(x^{[j,a]}\right) \qquad \forall\, x \in I^{d-1}$$

and

$$g_{j,a}(x) = \left[g_2(x^{[j,a]}),\ldots,g_d(x^{[j,a]})\right], \qquad \forall\, x \in I^{d-1},$$

where

$$x^{[j,a]} = [x_1,\ldots,x_{j-1},a,x_{j+1},\ldots,x_d].$$

Note that

$$\|g_{1,j,a}\|_{C^1(I^{d-1};\mathbb{R}^{d-1})} \le \|g\|_{C^1(I^d;\mathbb{R}^d)} \tag{4.3}$$

and

$$\|g_{j,a}\|_{C^1(I^{d-1};\mathbb{R}^{d-1})} \le \|g\|_{C^1(I^d;\mathbb{R}^d)}. \tag{4.4}$$

We have the following

Lemma 4.1. *If* $g \in C^1(I^d;\mathbb{R}^d)$, *then*

$$S_d(g) = \sum_{j=1}^{d}(-1)^{j+1}\left[\tilde{S}_{d-1}(g_{1,j,1}, g_{j,1}) - \tilde{S}_{d-1}(g_{1,j,0}, g_{j,0})\right].$$

Proof. Suppose first that $g \in C^s(I^d;\mathbb{R}^d)$, where $s \ge 2$. Then [5, Chapter 4, Theorem 3.2] states that we can write $\det \nabla g$ in divergence form as

$$(\det \nabla g)(x) = \sum_{j=1}^{d}(-1)^{j+1}\partial_j\big(g_{1,j,x_j}(x)(\det \nabla g_{j,x_j})(x)\big) \qquad \forall\, x \in I^{d-1}. \tag{4.5}$$

Integrating by parts, we see that the lemma holds for $s \ge 2$.

Now suppose that $g \in C^1(I^d;\mathbb{R}^d)$. We use a density argument to show that the lemma holds for this case. Indeed, for any $\delta > 0$ and any index $i \in \{1,\dots,d\}$, we can find a function $p_{i,\delta} \in C^2(I^d;\mathbb{R})$ such that $\|g_i - p_{i,\delta}\|_{C^1(I^d;\mathbb{R})} \le \delta$. Let $p_\delta = [p_{1,\delta},\dots,p_{d,\delta}]$. Since $p_\delta \in C^2(I^d;\mathbb{R}^d)$, we have

$$S_d(p_\delta) = \sum_{j=1}^{d}(-1)^{j+1}\left[\tilde{S}_{d-1}\big((p_\delta)_{1,j,1}, (p_\delta)_{j,1}\big) - \tilde{S}_{d-1}\big((p_\delta)_{1,j,0}, (p_\delta)_{j,0}\big)\right]$$

Now let δ tend to zero. Since S_d and $\tilde{S}_{d-1}$ are continuous, we now see that the lemma holds when $s = 1$. □

The essence of Lemma 4.1 is that the d-dimensional signed volume problem is equal to the sum of $2d$ instances of $(d-1)$-dimensional integrals of the form (4.2). These latter integrals are similar to (but simpler than) the integrals

$$\int_{I^{d-1}} f\big(w(x)\big)(\det \nabla w)(x)\,dx$$

studied in [12]. We can apply the analysis of [12] to handle such problems in the case $s \ge 2$.

Hence, we shall consider two separate cases.

4.1 The case $d \geq 2$ and $s \geq 2$

Let $g \in G_s$. Our approximation $U_n(g)$ to $S_d(g)$ will have the form

$$U_{d,n}(g) = \sum_{j=1}^{d} (-1)^{j+1} \big(U_{d-1,n;\,j,1}(g) - U_{d-1,n;\,j,0}(g) \big), \tag{4.6}$$

where $U_{d-1,n;\,j,a}(g)$ is an approximation of $\tilde{S}_{d-1}(g_{1,j,a}, g_{j,a})$ for any $a \in \{0, 1\}$ and any $j \in \{1, \ldots, d\}$. From Lemma 4.1 and (4.6), it follows that

$$\begin{aligned} S_d(g) - U_{d,n}(g) = \sum_{j=1}^{d} (-1)^{j+1} \Big[\big(\tilde{S}_{d-1}(g_{1,j,1}, g_{j,1}) - U_{d-1,n;\,j,1}(g) \big) - \\ \big(\tilde{S}_{d-1}(g_{1,j,0}, g_{j,0}) - U_{d-1,n;\,j,0}(g) \big) \Big]. \end{aligned} \tag{4.7}$$

We briefly describe the approximation $U_{d-1,n;\,j,a}$ that appears in (4.6). Define $\mu = \max\{s-1, 2\}$. Let $\mathscr{Q}$ be a uniform decomposition of the face $x_j = a$ of I^d, with a meshsize proportional to $n^{-1/(d-1)}$. Next, we let $\mathscr{S}$ be a globally $C^{\mu-1}$ tensor product spline space of degree μ corresponding to $\mathscr{Q}$. For $g \in G_s$, let $\overline{g} \in \mathscr{S}$ be an appropriately-chosen quasi-interpolant of g that can be computed using n function values of g. Then we take $U_{d-1,n;\,j,a}(g) = \tilde{S}_{d-1}(\overline{g}_{1,j,a}, \overline{g}_{j,a})$. For more details, see [12].

Lemma 4.2. *If $s \geq 2$, then the approximation $U_{d,n}$ defined by* (4.6) *satisfies*

$$\operatorname{cost} U_{d,n} \preccurlyeq \boldsymbol{c}\, n$$

and

$$e(U_{d,n}; S_d) \preccurlyeq \left(\frac{1}{n} \right)^{s/(d-1)}.$$

Proof. Let $j \in \{1, \ldots, d\}$ and $a \in \{0, 1\}$. Using a straightforward adaption of the techniques of [12], which required that $s \geq 2$, along with the bounds (4.3)–(4.4), we find that

$$\operatorname{cost} U_{d-1,n;\,j,a} \preccurlyeq \mathbf{c}\, n$$

and

$$\sup_{g \in G_s} |\tilde{S}_{d-1}(g_{1,j,a}, g_{j,a}) - U_{d-1,n;\,j,a}(g)| \preccurlyeq \left(\frac{1}{n} \right)^{s/(d-1)}.$$

The lemma follows immediately from these bounds and from (4.7). □

Combining Theorem 4.1 with Lemma 4.2, and recalling the comments at the end of Section 2, we have the main result of this section:

Theorem 4.2. *If $d \geq 2$ and $s \geq 2$, then*

$$e(n; S) \asymp \left(\frac{1}{n}\right)^{s/(d-1)}$$

and

$$\mathrm{comp}(\varepsilon; S) \asymp \mathbf{c} \left(\frac{1}{\varepsilon}\right)^{(d-1)/s}.$$

Moreover, let U_n be given by Lemma 4.2. Then the approximation $|U_n|$ of the volume operator S, with $n \asymp (1/\varepsilon)^{(d-1)/s}$, is optimal.

4.2 The case $d \geq 2$ and $s = 1$

The minimal-smoothness case $s = 1$ must be handled more delicately than the preceding case $s \geq 2$. For a meshsize $h = 1/m$, with m a positive integer, we shall let $U_{d,h}$ and $\tilde{U}_{d,h}$ denote approximations to S_d and $\tilde{S}_d$, respectively. These approximations will be defined by induction on d.

We first handle the case $d = 1$, defining

$$U_{1,h}(g) = g(1) - g(0) \tag{4.8}$$

and

$$\tilde{U}_{1,h}(f, w) = h \sum_{i=0}^{m-1} f(ih)\big[w\big((i+1)h\big) - w(ih)\big].$$

Note that

$$\mathrm{cost}\, U_{1,h} = 2\,\mathbf{c} + 1$$

and

$$\mathrm{cost}\, \tilde{U}_{1,h} \asymp \mathbf{c}\, \frac{1}{h}.$$

Clearly

$$S_1(g) = U_{1,h}(g) \tag{4.9}$$

and

$$\begin{aligned} \tilde{S}_{1,h}(f, w) - \tilde{U}_{1,h}(f, w) &= \sum_{i=0}^{m-1} \int_{ih}^{(i+1)h} [f(x) - f(ih)] w'(x)\, dx \\ &= \sum_{i=0}^{m-1} f'(\xi_{i,h}) h \int_{ih}^{(i+1)h} w'(x)\, dx \end{aligned}$$

where $\xi_{i,h} \in \big(ih, (i+1)h\big)$ for $i \in \{0, \ldots, m-1\}$. From this we conclude that

$$|\tilde{S}_1(f, w) - \tilde{U}_{1,h}(f, w)| \leq h \|f\|_{C^1(I;\mathbb{R})} \|w\|_{C^1(I;\mathbb{R})}. \tag{4.10}$$

For $d \geq 2$, we will use induction on d to define $U_{d,h}$ and $\tilde{U}_{d,h}$. First, we need to express S_d and $\tilde{S}_d$ in terms of S_{d-1} and $\tilde{S}_{d-1}$. For S_d, this has already been established in Lemma 4.1. Hence, we focus our attention on $\tilde{S}_d$. Since the w appearing in $\tilde{S}_d(f, w)$ is only a C^1-function, we see that $\det \nabla w$ is merely continuous, and cannot be approximated to within an arbitrary error by using finitely many samples of w. Our only hope is to reduce the dimension in $\tilde{S}_d$. As we know, this could be done if f were a constant. Since this is not the case, we shall approximate f by a piecewise constant function over small subcubes of I^d, and then use Lemma 4.1 to reduce the dimension.

More precisely, let

$$K_{\alpha,h} = \{\, h(\alpha + t) : t \in I^d \,\}$$

be a subcube, with $\alpha \in M_h = \{0, \ldots, m-1\}^d$, where $h = 1/m$. Then

$$\tilde{S}_d(f, w) = \sum_{\alpha \in M_h} \int_{K_{\alpha,h}} f(x)(\det \nabla w)(x)\, dx.$$

Let

$$e_h(f, w) = \tilde{S}_d(f, w) - \sum_{\alpha \in M_h} f(h\alpha) \int_{K_{\alpha,h}} (\det \nabla w)(x)\, dx.$$

Since

$$|(\det \nabla w)(x)| \preccurlyeq \|w\|^d_{C^1(I^d;\mathbb{R}^d)} \qquad \forall\, x \in I^d,$$

it is easy to see that

$$|e_h(f, w)| \preccurlyeq h \|f\|_{C^1(I^d)} \|w\|^d_{C^1(I^d)}. \tag{4.11}$$

Define

$$w_{\alpha,h}(t) = \frac{w\big(h(\alpha + t)\big) - w(h\alpha)}{h} \qquad \forall\, t \in I^d. \tag{4.12}$$

Then the chain rule yields

$$\|w_{\alpha,h}\|_{C^1(I^d;\mathbb{R}^d)} \leq \|w\|_{C^1(I^d;\mathbb{R}^d)} \qquad \forall\, \alpha \in M_h \tag{4.13}$$

and a change of variables $x = h(\alpha + t)$ yields

$$\int_{K_{\alpha,h}} (\det \nabla w)(x)\, dx = h^d \int_{K_{\alpha,h}} (\det \nabla w_{\alpha,h})(t)\, dt = h^d S_d(w_{\alpha,h}).$$

Hence

$$\tilde{S}_d(f, w) = h^d \sum_{\alpha \in M_h} f(h\alpha) S_d(w_{\alpha,h}) + e_h(f, w). \tag{4.14}$$

From Lemma 4.1 and (4.14), we see how to reduce the dimension d. These formulas also suggest that for $d \geq 2$, we should define $U_{d,h}$ as

$$U_d(g) = \sum_{j=1}^{d} (-1)^{j+1} \left[\tilde{U}_{d-1}(g_{1,j,1}, g_{j,1}) - \tilde{U}_{d-1}(g_{1,j,0}, g_{j,0}) \right] \tag{4.15}$$

and $\tilde{U}_{d,h}$ as

$$\tilde{U}_d(f, w) = h^d \sum_{\alpha \in M_h} f(h\alpha) U_{d,h}(w_{\alpha,h}). \tag{4.16}$$

We then have the following

Lemma 4.3. *1. For $f \in C^1(I^d; \mathbb{R})$ and $w \in C^1(I^d; \mathbb{R}^d)$, we have*

$$\operatorname{cost} \tilde{U}_{d,h} \preccurlyeq \mathbf{c} \left(\frac{1}{h} \right)^{d(d+1)/2}$$

and

$$|\tilde{S}_d(f, w) - \tilde{U}_{d,h}(f, w)| \preccurlyeq h \cdot \|f\|_{C^1(I^d;\mathbb{R})} \|w\|^d_{C^1(I^d;\mathbb{R}^d)}.$$

2. For $g \in C^1(I^d; \mathbb{R}^d)$, we have

$$\operatorname{cost} U_{d,h} \preccurlyeq \mathbf{c} \left(\frac{1}{h} \right)^{(d-1)d/2}$$

and

$$|S_d(g) - U_{d,h}(g)| \preccurlyeq h \cdot \|g\|^d_{C^1(I^d;\mathbb{R}^d)}.$$

Proof. The proof is by induction on d. For $d = 1$, the result follows from (4.9) and (4.10). So, we proceed to the case $d \geq 2$.

Let us first estimate the costs of $U_{d,h}$ and $\tilde{U}_{d,h}$. We estimate the cost of $U_{d,h}$ from (4.15). Note that the evaluation of $w_{\alpha,h}$ requires at most two evaluations of w and two arithmetic operations. Thus we have

$$\operatorname{cost} U_{d,h} \leq 2d \operatorname{cost} \tilde{U}_{d-1,h} + d \preccurlyeq \left(\frac{1}{h} \right)^{d-1} \operatorname{cost} U_{d-1,h} + \mathbf{c} \left(\frac{1}{h} \right)^{d}.$$

By induction, we obtain

$$\operatorname{cost} U_{d,h} \preccurlyeq \mathbf{c} \left(\frac{1}{h} \right)^{(d-2)(d-1)/2+(d-1)} = \mathbf{c} \left(\frac{1}{h} \right)^{(d-1)d/2},$$

Similarly, we get from (4.16) that

$$\operatorname{cost} \tilde{U}_{d,h} \preccurlyeq \left(\frac{1}{h} \right)^{d} \operatorname{cost} U_{d,h} + \mathbf{c} \left(\frac{1}{h} \right)^{d} \preccurlyeq \mathbf{c} \left(\frac{1}{h} \right)^{(d-1)d/2+d} = \mathbf{c} \left(\frac{1}{h} \right)^{d(d+1)/2},$$

as required.

It only remains to estimate the errors of $U_{d,h}$ and $\tilde{U}_{d,h}$ for $d \geq 2$. From Lemma 4.1 and (4.15), we have

$$S_d(g) - U_d(g) = \sum_{j=1}^{d} (-1)^{j+1} \Big(\big[\tilde{S}_{d-1}(g_{1,j,1}, g_{j,1}) - \tilde{U}_{d-1}(g_{1,j,1}, g_{j,1}) \big] - \big[\tilde{S}_{d-1}(g_{1,j,0}, g_{j,0}) - \tilde{U}_{d-1}(g_{1,j,0}, g_{j,0}) \big] \Big). \quad (4.17)$$

Using the induction hypothesis, we obtain

$$|S_d(g) - U_{d,h}(g)| \preccurlyeq h \sum_{j=1}^{d} \Big(\|g_{1,j,1}\|_{C^1(I^{d-1};\mathbb{R}^{d-1})} \|g_{j,1}\|^{d-1}_{C^1(I^{d-1};\mathbb{R}^{d-1})} + \|g_{1,j,0}\|_{C^1(I^{d-1};\mathbb{R}^{d-1})} \|g_{j,0}\|^{d-1}_{C^1(I^{d-1};\mathbb{R}^{d-1})} \Big).$$

From (4.3), (4.4), and this inequality, we get

$$|S_d(g) - U_{d,h}(g)| \preccurlyeq h \|g\|^{d}_{C^1(I^d;\mathbb{R}^d)}, \quad (4.18)$$

as claimed. Similarly, (4.14) and (4.16) yield

$$\tilde{S}_d(f, w) - \tilde{U}_d(f, w) = h^d \sum_{\alpha \in M_h} f(h\alpha)[S_d(w_{\alpha,h}) - U_{d,h}(w_{\alpha,h})] + e_h(f, w).$$

From (4.11) and (4.18) with g replaced by $w_{\alpha,h}$, we obtain

$$|\tilde{S}_d(f, w) - \tilde{U}_d(f, w)| \preccurlyeq h^d \|f\|_{C^1(I^d;\mathbb{R})} \sum_{\alpha \in M_h} h \|w_{\alpha,h}\|^{d}_{C^1(I^d;\mathbb{R}^d)}.$$

Using (4.13), we finally obtain

$$|\tilde{S}_d(f, w) - \tilde{U}_d(f, w)| \preccurlyeq h^{d+1} \|f\|_{C^1(I^d;\mathbb{R})} \cdot \frac{1}{h^d} \|w\|^{d}_{C^1(I^d;\mathbb{R}^d)} = h \|f\|_{C^1(I^d;\mathbb{R})} \|w\|^{d}_{C^1(I^d;\mathbb{R}^d)},$$

as required to complete the proof of the Lemma. □

Observe that for $d = 2$, Lemma 4.3 tells us that $\operatorname{cost} U_{d,1/n} \preccurlyeq \mathbf{c}\, n$ and that $e(U_{d,1/n}) \preccurlyeq 1/n$. By Theorem 4.1, these are optimal. For $d \geq 3$, there is no such match between the lower bounds of Theorem 4.1 and the upper bounds of Lemma 4.3. We only know that $\operatorname{cost} U_{d,h} \preccurlyeq \mathbf{c}\, n$ and $e(U_{d,h}) \preccurlyeq (1/n)^{2/((d-1)d)}$ when $h = (1/n)^{2/((d-1)d)}$. Recalling the comments at the end of Section 2, and using Theorem 4.1 with Lemmas 4.2 and 4.3, we have the main result of this section:

Theorem 4.3. *Let $s = 1$ and $d \geq 2$.*

1. *If $d = 2$, then*
$$e(n; S) \asymp \frac{1}{n}$$
and
$$\mathrm{comp}(\varepsilon; S) \asymp \boldsymbol{c}\,\frac{1}{\varepsilon}.$$
Moreover, let $U_{d,h}$ be given by Lemma 4.3. Then the approximation $|U_{d,h}|$ of the volume operator S, with $h \asymp \varepsilon$, is optimal.

2. *If $d \geq 3$, then*
$$\left(\frac{1}{n}\right)^{1/(d-1)} \preccurlyeq e(n; S) \preccurlyeq \left(\frac{1}{n}\right)^{2/((d-1)d)}$$
and
$$\left(\frac{1}{\varepsilon}\right)^{d-1} \preccurlyeq \mathrm{comp}(\varepsilon; S) \preccurlyeq \boldsymbol{c}\left(\frac{1}{\varepsilon}\right)^{d(d-1)/2}.$$
Moreover, let $U_{d,h}$ be given by Lemma 4.3. Then these two upper bounds are attained by the approximations $|U_{d,h}|$, with $h = (1/n)^{2/((d-1)d)}$ and $h \asymp \varepsilon$, respectively.

Note that we only know that the result of Theorem 4.3 is optimal when $d = 2$. Determining tight bounds on the complexity of volume calculation for $s = 1$ is an open question for the case $d \geq 3$.

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