Tractability of Quasilinear Problems I: General Results*

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Abstract

The tractability of multivariate problems has usually been studied only for the approximation of linear operators. In this paper we study the tractability of quasilinear multivariate problems. That is, we wish to approximate nonlinear operators $S_d(\cdot,\cdot)$ that depend linearly on the first argument and satisfy a Lipschitz condition with respect to both arguments. Here, both arguments are functions of d variables. Many computational problems of practical importance have this form. Examples inlude the solution of specific Dirichlet, Neumann, and Schrödinger problems. We show, under appropriate assumptions, that quasilinear problems, whose domain spaces are equipped with product or finite-order weights, are tractable or strongly tractable in the worst case setting.

This paper is the first part in a series of papers. Here, we present tractability results for quasilinear problems under general assumptions on quasilinear operators and weights. In future papers, we shall verify these assumptions for quasilinear problems such as the solution of specific Dirichlet, Neumann, and Schrödinger problems.

1 Introduction

The tractability of multivariate problems has recently become an extensive research area; see [7] for a survey. For such problems, we wish to approximate operators S_d defined over classes of functions g of d variables, where d may be very large. Such problems occur in computational practice. Probably the best-known

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source of such problems is mathematical finance, wherein applications are known for which d can be in the hundreds or even in the thousands and S_d is a linear integration operator; see [13] and references cited there for examples.

One goal of tractability studies is to prove that the minimal number of evaluations of g needed to approximate $S_d(g)$ to within ε is polynomial in ε^{-1} and d; see [17]. In most tractability papers, it is assumed that S_d is a *linear* operator.[1] A typical result for linear operators is that as long as we consider isotropic spaces (in which all d variables play the same role), then tractability does *not* hold[2] since the minimal number of evaluations is exponential in d. This is called the *curse of dimensionality*. To break intractability or the curse of dimensionality, we may treat variables or groups of variables of linear multivariate problems in a non-isotropic way. This leads to *weighted* spaces of functions; the paper [11] is probably the first to study this idea. For many linear multivariate problems (including, e.g., integration and approximation), we know conditions on the weights that are necessary and sufficient for tractability; see again [7] for a survey.

Tractability has been studied for several kinds of weights; see, e.g., [3] for further discussion. The first papers dealt with *product* weights, where the jth variable was moderated by a specific weight γ_j. A typical result is that tractability holds iff $\sum_{j=1}^{d} \gamma_j$ is bounded by a multiple of $\ln d$. Hence the isotropic case, for which $\gamma_j = 1$, is intractable. On the other hand, suppose we have a decreasing polynomial dependence on the successive variables, so that $\gamma_j = \Theta(j^{-\alpha})$. Then the problem is tractable iff $\alpha \geq 1$. For $\alpha > 1$, the series $\sum_{j=1}^{\infty} \gamma_j$ is convergent, and we often have *strong* tractability. That is, the minimal number of evaluations of the function g to approximate $S_d(g)$ to within ε does *not* depend on d and is polynomially bounded in ε^{-1}.

The second class of weights is the class of *finite-order* weights, which has recently been studied, see [3] where finite-order weights were first defined and [3, 10, 14] where finite-order weights were further studied. Such weights are used to model functions of d variables that can be represented as, or approximated by, a sum of functions of fewer variables. That is, each term of this sum depends on at most ω variables, with ω independent of d. It turns out that finite-order weights imply tractability, or even strong tractability, for many linear multivariate problems even in the worst case.

The purpose of this paper is to extend the study of tractability to certain nonlinear multivariate problems. We restrict ourselves to *quasilinear* multivariate problems. That is, we wish to approximate $S_d(f, q)$, where

1. f and q are d-variate functions,
2. $S_d(f, q)$ depends linearly on f, and
3. $S_d(f, q)$ satisfies a Lipschitz condition with respect to both f and q.

Many computational problems of practical importance have this form. Examples include the solution of specific Dirichlet, Neumann and Schrödinger differential equations. These problems are roughly defined as follows. Let $I^d = (0, 1)^d$, and let f and q be functions defined over I^d, enjoying given smoothness properties, with q being non-negative.

1. The *Dirichlet problem* defines $u = S_d(f, q)$ as the variational solution of the Poisson equation

$$-\Delta u + qu = f \qquad \text{in } I^d,$$

subject to homogeneous Dirichlet boundary conditions

$$u = 0 \qquad \text{on } \partial I^d.$$

[1] The only exception of which we are aware is the paper [9], where tractability of fixed points for economics problems is studied.

[2] There are, however, at least two examples, where tractability holds for isotropic spaces; see [5, 6].

2. The *Neumann problem* defines $u = S_d(f, q)$ as the variational solution of the Poisson equation given above, subject to homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial I^d,$$

with $\partial/\partial\nu$ denoting the outer-directed normal derivative.

3. The *Schrödinger problem* defines $u = S_d(f, q)$ as the variational solution of the Schrödinger equation

$$i\frac{\partial u}{\partial t} = -\Delta u + qu \qquad \text{in } I^d$$

for $t > 0$, with the initial condition

$$u(\cdot, 0) = f,$$

subject to homogeneous Dirichlet boundary conditions.

For these problems, the function f corresponds to the right hand side of the differential equation or the initial value of the solution, whereas the function q is part of the differential operator. Then the solution $S_d(f, q)$ depends linearly on f and nonlinearly on q, and has Lipschitz dependence on both f and q.

We study quasilinear problems for weighted spaces. Our main emphasis is on product and finite-order weights. We show that the tractability results of [14] for the approximation problem can be extended to quasilinear problems. We obtain tractability, or even strong tractability, of quasilinear problems, under appropriate assumptions on quasilinear problems and weights.

This paper is the first in a series of papers. Here we present tractability results for general quasilinear problems under certain assumptions on the operators S_d and the weights. We shall show in future papers that these assumptions hold for quasilinear problems such as the solution of specific Dirichlet, Neumann and Schrödinger problems.

We now discuss the approach in this paper in more technical terms. Let $g = (f, q)$. We approximate $S_d(g)$ by algorithms evaluating finitely many functionals of f and q. The form of these functionals is restricted to a specific class Λ. We consider two classes Λ. The first class consists of all continuous linear functionals, and the second class consists of only function values. We define the error of an algorithm in the worst case setting. We consider two error criteria: absolute and normalized. For the absolute error criterion, we want to find an algorithm whose worst case error is at most ε; for the normalized error criterion, we want to find an algorithm whose worst case error reduces the initial error by a factor ε. Here, the initial error is defined as the minimal worst case error over algorithms using no evaluations of g. In both cases, we say that the algorithm computes an ε-approximation to the operator S_d.

Let $\mathrm{card}(\varepsilon, S_d, \Lambda)$ denote the minimal number of evaluations from the class Λ needed to find an ε-approximation of the operator S_d under the given error criterion. The problem is tractable if $\mathrm{card}(\varepsilon, S_d, \Lambda)$ depends polynomially on ε^{-1} and d, and is strongly tractable if $\mathrm{card}(\varepsilon, S_d, \Lambda)$ is bounded independently of d by a polynomial in ε^{-1}. Using the results and proof techniques of [14], we present several estimates of $\mathrm{card}(\varepsilon, S_d, \Lambda)$. For product weights and finite-order weights, we prove tractability and strong tractability of general quasilinear problems, under appropriate assumptions.

The main idea behind our approach is that we use the results from [14] for the multivariate approximation problem. More precisely, we know from [14] that there are algorithms A using a polynomial number of evaluations in ε^{-1} and d such that $A(f)$ and $A(q)$ are ε-approximations of f and q, respectively. We then approximate $S_d(f, q)$ by $S_d(A(f), A(q))$. We underline that the results of [14] are constructive for the class of all continuous linear functionals, and non-constructive for the class of function values. Therefore our

results are also non-constructive for the second class. To overcome this problem, one could use the results of the recent paper [15], which contains constructive results for the multivariate approximation problem for the class of function values, with error bounds that are sometimes slightly worse. In this way, one can obtain constructive results for quasilinear problems and the class of function values.

Finally, we want to stress that so far we have studied tractability of quasilinear problems only in terms of the number of functionals needed to obtain an ε-approximation. We have not considered the problem of how many arithmetic operations are needed to implement the algorithms for which we obtained the tractability bounds. This problem is easier for linear multivariate problems, since it is enough to consider linear algorithms requiring precomputation of as many elements as are found in the tractability bounds. However, for quasilinear problems, we use nonlinear algorithms since we need to compute $S_d(\tilde{f}, \tilde{q})$ for known $\tilde{f} = A(f)$ and $\tilde{g} = A(g)$. It is not clear a priori how to implement this at cost polynomial in ε^{-1} and d. We are, however, optimistic that this can be achieved at least for some quasilinear problems of practical importance. We will study this issue in the future.

2 Tensor products of RKHS with general weights

We first establish a few notational conventions. If R is an ordered ring, then R^+ and R^{++} respectively denote the non-negative and positive elements of R. If X and Y are normed linear spaces, then $\mathrm{Lin}[X, Y]$ denotes the space of bounded linear transformations of X into Y. We write $\mathrm{Lin}[X]$ for $\mathrm{Lin}[X, X]$, and X^* for $\mathrm{Lin}[X, \mathbb{R}]$. Finally, we use the standard notation for Sobolev inner products, seminorms, norms, and spaces, found in, e.g., [8, 16].

We first discuss the univariate case. For $I = (0, 1)$, let $K : \bar{I} \times \bar{I} \to \mathbb{R}$ be a nonzero, symmetric, positive definite function, i.e., the matrix $[K(x_i, x_j)]_{i,j=1}^n$ is a positive semidefinite matrix for any $n \in \mathbb{Z}^{++}$ and any distinct $x_1, \ldots, x_n \in \bar{I}$. We assume that

$$\kappa_1 := \int_0^1 K(x, x)\, dx < \infty. \tag{1}$$

Let $H(K)$ be the reproducing kernel Hilbert space (RKHS) associated with the kernel K, so that

$$f(x) = \langle f, K(\cdot, x) \rangle_{H(K)} \qquad \forall\, x \in \bar{I},\ f \in H(K).$$

As in [14, Sect. 2], we find that

$$\|f\|_{L_2(I)} := \left(\int_0^1 \big(f(x)\big)^2\, dx \right)^{1/2} \le \kappa_1^{1/2} \|f\|_{H(K)} \qquad \forall f \in H(K),$$

so that $H(K)$ is embedded in $L_2(I)$.

We now turn to the multivariate case. For $d \in \mathbb{Z}^{++}$, let $\mathscr{P}_d$ denote the power set of $\{1, 2, \ldots, d\}$. Let

$$\gamma = \{\, \gamma_{d,\mathfrak{u}} : \mathfrak{u} \in \mathscr{P}_d,\, d \in \mathbb{Z}^{++} \,\}$$

be a set of non-negative weights. If we denote the cardinality of a set by $|\cdot|$, then obviously

$$\big|\{\, \gamma_{d,\mathfrak{u}} : \mathfrak{u} \in \mathscr{P}_d \,\}\big| \le 2^d \qquad \forall\, d \in \mathbb{Z}^{++}.$$

The most well-studied examples of such weights are the following (see, e.g., [3]):

1. We say that γ is a set of *product* weights if there exist numbers $\gamma_1 \geq \gamma_2 \geq \cdots \geq 0$ such that

$$\gamma_{d,\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j \qquad \forall\, \mathfrak{u} \in \mathscr{P}_d,\ d \in \mathbb{Z}^{++}.$$

2. We say that γ is a set of *finite-order* weights if for some $\omega \in \mathbb{Z}^{++}$, we have

$$\gamma_{d,\mathfrak{u}} = 0 \qquad \forall\, \mathfrak{u} \in \mathscr{P}_d \text{ and } |\mathfrak{u}| > \omega,\ d \in \mathbb{Z}^{++}. \tag{2}$$

   The *order* of a set γ of finite-order weights is the smallest $\omega \in \mathbb{Z}^{++}$ such that (2) holds.

For $d \in \mathbb{Z}^{++}$ and $\mathfrak{u} \in \mathscr{P}_d$, define $K_{d,\mathfrak{u}}\colon \bar{I}^d \times \bar{I}^d \to \mathbb{R}$ as

$$K_{d,\mathfrak{u}}(\mathbf{x}, \mathbf{y}) = \prod_{j \in \mathfrak{u}} K(x_j, y_j) \qquad \forall\, \mathbf{x}, \mathbf{y} \in \bar{I}^d.$$

We then let $H(K_{d,\mathfrak{u}})$ be the RKHS with reproducing kernel $K_{d,\mathfrak{u}}$. By convention,

$$K_{d,\emptyset} = 1 \qquad \text{and} \qquad H(K_{d,\emptyset}) = \text{span}\{1\}.$$

For nonempty $\mathfrak{u}$, the space $H(K_{d,\mathfrak{u}})$ is the tensor product space of the spaces of univariate functions with indices from the set $\mathfrak{u}$.

Let γ be a weight sequence. As in [14], for $d \in \mathbb{Z}^{++}$, let $H(K_d)$ be the RKHS whose reproducing kernel is

$$K_d = \sum_{\mathfrak{u} \in \mathscr{P}_d} \gamma_{d,\mathfrak{u}} K_{d,\mathfrak{u}}.$$

For $f \in H(K_d)$, we can write

$$f = \sum_{\mathfrak{u} \in \mathscr{P}_d} f_{\mathfrak{u}}, \qquad \text{where } f_{\mathfrak{u}} = \gamma_{d,\mathfrak{u}} f_{d,\mathfrak{u}} \in H(K_{d,\mathfrak{u}}). \tag{3}$$

The term $f_{d,\mathfrak{u}}$ in this decomposition depends on the $|\mathfrak{u}|$ variables indexed by $\mathfrak{u}$. For weights of order ω, the sum consists of $O(d^\omega)$ terms, with each term consisting of at most ω variables.

Since the decomposition (3) is generally not unique, we have

$$\|f\|^2_{H(K_d)} = \inf \sum_{\mathfrak{u} \in \mathscr{P}_d} \gamma_{d,\mathfrak{u}} \|f_{d,\mathfrak{u}}\|^2_{H(K_{d,\mathfrak{u}})} \qquad \forall f \in H(K_d),$$

the infimum being taken over all $\{f_{d,\mathfrak{u}} \in H(K_{d,\mathfrak{u}})\}_{\mathfrak{u} \in \mathscr{P}_d}$ such that (3) holds; see [1] for further discussion.

The decomposition (3) is unique iff $1 \notin H(K)$, in which case we have the orthogonal direct sum decomposition

$$H(K_d) = \bigoplus_{\mathfrak{u} \in \mathscr{P}_d} H(K_{d,\mathfrak{u}}),$$

along with the explicit formula

$$\langle f, g \rangle_{H(K_d)} = \sum_{\mathfrak{u} \in \mathscr{P}_d} \gamma_{d,\mathfrak{u}} \langle f_{d,\mathfrak{u}}, g_{d,\mathfrak{u}} \rangle_{H(K_{d,\mathfrak{u}})} \qquad \forall f, g \in H(K_d)$$

for the $H(K_d)$-inner product.

Example. Let

$$\kappa_2 = \int_0^1 \int_0^1 K(x, y)\, dy\, dx. \tag{4}$$

Since K is a reproducing kernel, it easily follows that $0 \leq \kappa_2 \leq \kappa_1$. If the kernel K is strictly positive definite, then $\kappa_2 > 0$. On the other hand, if $\kappa_2 = 0$, then [14, Lemma 1] tells us that $1 \notin H(K)$, implying that we have the orthogonal direct sum decomposition given above. □

Define

$$\sigma_d(\theta) = \left(\sum_{\mathfrak{u} \in \mathscr{P}_d} \gamma_{d,\mathfrak{u}} \theta^{|\mathfrak{u}|} \right)^{1/2} \qquad \forall\, \theta \in \mathbb{R}^+. \tag{5}$$

Clearly, see also [14, Sect. 2], we have

$$\int_{\bar{I}^d} K_d(\mathbf{x}, \mathbf{x})\, d\mathbf{x} = \sigma_d^2(\kappa_1),$$

and

$$\|f\|_{L_2(I^d)} := \left(\int_{\bar{I}^d} \big(f(x)\big)^2\, dx \right)^{1/2} \leq \sigma_d(\kappa_1) \|f\|_{H(K_d)} \qquad \forall f \in H(K_d).$$

Hence $H(K_d)$ is embedded in $L_2(I^d)$.

3 Problem formulation

We consider operators

$$S_d \colon H(K_d) \times Q_d \to G_d,$$

where

1. G_d is a normed linear space, and
2. Q_d is a set of real-valued functions defined over I^d.

We require our problem to be *quasilinear*, meaning that S_d is linear with respect to the first argument, and satisfies a Lipschitz condition with respect to both arguments. The formal definition is given in Section 5.

For $d \in \mathbb{Z}^{++}$, define

$$H_{d,\rho} = \{\, f \in H(K_d) : \|f\|_{H(K_d)} \leq \rho \,\} \qquad \forall\, \rho > 0$$

as the ball in $H(K_d)$ of radius ρ. Our goal is to efficiently approximate $S_d(f, q)$ for $[f, q] \in H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2})$. Here, ρ_1 and ρ_2 are positive constants, which are independent of d, and we assume that $Q_d \cap H_{d,\rho_2}$ is nonempty.

Note that there is a certain lack of symmetry in our class $H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2})$ of problem elements. The first factor H_{d,ρ_1} is a ball in the space $H(K_d)$, whereas the second factor $Q_d \cap H_{d,\rho_2}$ is not a ball in a space, but is the intersection of such a ball with some other set of functions. This asymmetry is needed to model many important problems, such as the elliptic Dirichlet problem.

Example: The Dirichlet Problem. Let G_d be the standard Sobolev space $H_0^1(I^d)$, and let $Q_d = \{q \in L_\infty(I^d) : q \geq 0\}$. For $[f, q] \in H(K_d) \times Q_d \subset L_2(I^d) \times Q_d$, standard results [2, 4, 8] on elliptic boundary-value problems tell us that there exists a unique $u \in H_0^1(I^d)$ such that

$$\int_{\bar{I}^d} [\nabla u \cdot \nabla w + quw] = \int_{\bar{I}^d} fw \qquad \forall\, w \in H_0^1(I^d).$$

Of course, u is the variational solution of the Dirichlet problem of finding $u\colon \bar{I}^d \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u + qu &= f && \text{in } I^d, \\ u &= 0 && \text{on } \partial I^d. \end{aligned}$$

Hence, if we write $u = S_d(f, q)$, we see that we have an operator $S_d\colon H(K_d) \times Q_d \to G_d$, as above.

The Dirichlet problem is specified by two functions, f and q. To solve this problem computationally, we need to assume that both functions enjoy some degree of smoothness, and this is modeled by a proper choice of the space $H(K_d)$. Hence, we have $f, q \in H(K_d)$. We also need to normalize f and q, since the problem cannot be solved otherwise. Therefore we assume that $f \in H_{d,\rho_1}$ and $q \in H_{d,\rho_2}$ for some ρ_1 and ρ_2, which presumably will not be too large. Since the Dirichlet problem is not well defined for arbitrary q from $H(K_d)$ we need to guarantee that q is also non-negative. We therefore have $q \in Q_d \cap H_{d,\rho_2}$, as required in our class of problem elements. □

The Dirichlet problem illustrates a general situation for quasilinear problems. We know that $S_d(\cdot, q)$ is linear for each choice of $q \in Q_d$. Hence the assumption about the first factor H_{d,ρ_1} should come as no surprise, being typical when studying the complexity of linear problems; see [12, Sect. 4.5.1]. On the other hand, there are many important problems such that $S_d(f, \cdot)$ is *not* defined over a ball of arbitrary radius in a function space for $f \in H(K_d)$, but must be defined only over a set of functions satisfying an additional condition; again turning to the elliptic Dirichlet problem, the simplest example of such a condition is that q be non-negative. This explains the presence of $Q_d \cap H_{d,\rho_2}$ in our definition.

We approximate $S(f, q)$ by computing finitely many values $\lambda(f)$ and $\lambda(q)$, where $\lambda \in \Lambda$. Here, Λ is a class of linear functionals on $H(K_d)$. We will restrict our attention to the following two choices:

1. $\Lambda^{\text{all}} = [H(K_d)]^*$, the set of all continuous linear functionals on $H(K_d)$. That is, $\lambda \in \Lambda^{\text{all}}$ iff there exists $r_\lambda \in H(K_d)$ such that

   $$\lambda(f) = \langle r_\lambda, f \rangle_{H(K_d)} \qquad \forall f \in H(K_d).$$

   Obviously,

   $$\|\lambda\|_{[H(K_d)]^*} = \|r_\lambda\|_{H(K_d)} \qquad \forall\, \lambda \in \Lambda^{\text{all}}.$$

2. Λ^{std}, the set of all function evaluations over $H(K_d)$. That is, $\lambda \in \Lambda^{\text{std}}$ iff there exists $\mathbf{x}_\lambda \in \bar{I}^d$ such that

   $$\lambda(f) = f(\mathbf{x}_\lambda) = \langle f, K_d(\cdot, \mathbf{x}_\lambda) \rangle_{H(K_d)} \qquad \forall f \in H(K_d).$$

   Clearly, we now have

   $$\|\lambda\|_{[H(K_d)]^*} = K_d^{1/2}(\mathbf{x}_\lambda, \mathbf{x}_\lambda) \qquad \forall\, \lambda \in \Lambda^{\text{std}},$$

   and $\Lambda^{\text{std}} \subset \Lambda^{\text{all}}$.

For $d \in \mathbb{Z}^{++}$ and $n \in \mathbb{Z}^{++}$, let $A_{d,n}$ be an algorithm for approximating S_d, using at most n information evaluations from Λ. That is,

$$A_{d,n}(f, q) = \phi(\lambda_1(f), \ldots, \lambda_k(f), \lambda_{k+1}(q), \ldots, \lambda_n(q))$$

for some $k \in [0, n]$, some $\lambda_1, \ldots, \lambda_n \in \Lambda$, and some mapping $\phi : \mathbb{R}^n \to G_d$. The linear functionals $\lambda_1, \ldots, \lambda_n$ can be chosen adaptively, along with the number n of functionals used; see, e.g., [12].

The worst case *error* of $A_{d,n}$ is defined to be

$$e(A_{d,n}, S_d, \Lambda) = \sup_{[f,q] \in H_{d,\rho_1} \times Q_d \cap H_{d,\rho_2}} \|S_d(f, q) - A_{d,n}(f, q)\|_{G_d}.$$

The nth *minimal error* is defined to be

$$e(n, S_d, \Lambda) = \inf_{A_{d,n}} e(A_{d,n}, S_d, \Lambda),$$

the infimum being over all algorithms using at most n information evaluations from Λ.

For $n = 0$ we do not use any information evaluations on f and q, and algorithms $A_{d,0}$ are just constant elements from G_d. Their worst case error is defined as above. The minimal error $e(0, S_d, \Lambda)$ is called the *initial error*. Since this initial error involves no information evaluations, it is independent of Λ, and hence we shall simply denote it as $e(0, S_d)$. From the results of [12, Sect. 4.5], we see that

$$\text{the problem is quasilinear} \implies e(0, S_d) = \rho_1 \sup_{q \in Q_d \cap H_{d,\rho_2}} \|S_d(\cdot, q)\|_{\mathrm{Lin}[H(K_d), G_d]}. \tag{6}$$

Let $\varepsilon \in (0, 1)$. We wish to measure the minimal number of information evaluations needed to compute an ε-approximation. Here, we say that an algorithm $A_{d,n}$ provides an *ε-approximation* to S_d if

$$e(A_{d,n}, S_d, \Lambda) \le \varepsilon \cdot \mathrm{ErrCrit}(S_d),$$

with ErrCrit being an *error criterion*. In this paper, we will use the error criteria

$$\mathrm{ErrCrit}(S_d) = \begin{cases} 1 & \text{for absolute error,} \\ e(0, S_d) & \text{for normalized error.} \end{cases}$$

Hence:

1. An algorithm provides an ε-approximation in the *absolute* sense simply means that the error of the algorithm is at most ε.

2. An algorithm provides an ε-approximation in the *normalized* sense simply means that the algorithm *reduces* the initial error by at least a factor of ε, and is thus at most $\varepsilon \cdot e(0, S_d)$.

For these two error criteria, let

$$\mathrm{card}(\varepsilon, S_d, \Lambda) = \min\{\, n \in \mathbb{Z}^+ : e(n, S_d, \Lambda) \le \varepsilon \cdot \mathrm{ErrCrit}(S_d) \,\}$$

denote the minimal number of information evaluations from Λ needed to obtain an ε-approximation of S_d. Of course, the ε-cardinalities for the absolute and normalized criteria are related by the equation

$$\mathrm{card}^{\mathrm{nor}}(\varepsilon, S_d, \Lambda) = \mathrm{card}^{\mathrm{abs}}(\varepsilon \cdot e(0, S_d), S_d, \Lambda). \tag{7}$$

We are ready to define tractability as in [17]. The problem $S = \{S_d\}_{d\in\mathbb{Z}^{++}}$ is said to be *tractable* in the class Λ if there exist non-negative numbers C, $p_{\rm err}$, and $p_{\rm dim}$ such that

$$\mathrm{card}(\varepsilon, S_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\rm err}} d^{\,p_{\rm dim}} \qquad \forall\, \varepsilon \in (0,1), d \in \mathbb{Z}^{++}. \tag{8}$$

Any numbers $p_{\rm err} = p_{\rm err}(S, \Lambda)$ and $p_{\rm dim} = p_{\rm dim}(S, \Lambda)$ such that (8) holds are called ε- and *d-exponents of tractability*; these need not be uniquely defined. If $p_{\rm dim} = 0$ in (8), then the problem S is said to be *strongly tractable* in Λ, and

$$p_{\rm strong}(S, \Lambda) = \inf \left\{ p_{\rm err} \ge 0 : \exists\, C \ge 0 \text{ such that } \mathrm{card}(\varepsilon, S_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\rm err}} \forall\, \varepsilon \in (0,1), d \in \mathbb{Z}^{++} \right\}$$

is called the *exponent of strong tractability*.

We stress that tractability results for the absolute sense may differ from those for the normalized sense, since $e(0, S_d)$ may depend on d.

4 Some results for the approximation problem

We need to recall some results from [14] about the *approximation problem*, i.e., the problem of approximating the embedding operator $\mathrm{App}_d \colon H(K_d) \to L_2(I^d)$ defined by $\mathrm{App}_d\, f = f$ for $f \in H(K_d)$. We will use these results in Section 5.

Let $d \in \mathbb{Z}^{++}$. The operator $W_d = (\mathrm{App}_d)^*(\mathrm{App}_d) \in \mathrm{Lin}[H(K_d)]$ may be explicitly written as

$$W_d f = \sum_{\mathfrak{u}\in\mathscr{P}_d} \int_{I^d} \gamma_{d,\mathfrak{u}} K_{d,\mathfrak{u}}(\mathbf{x}, \cdot)\, f(\mathbf{x})\, d\mathbf{x} \qquad \forall f \in H(K_d).$$

We will also need to use the embedding operator $\mathrm{App} \in \mathrm{Lin}[H(K), L_2(I)]$, as well as the operator $W = (\mathrm{App})^*(\mathrm{App}) \in \mathrm{Lin}[H(K)]$. The latter is given explicitly as

$$Wf = \int_0^1 K(x, \cdot)\, f(x)\, dx \qquad \forall f \in H(K). \tag{9}$$

Since W_d is a self-adjoint compact operator on $H(K_d)$, there exist eigenvalues

$$\lambda_{d,1} \ge \lambda_{d,2} \ge \cdots \ge 0$$

and an orthonormal basis $\{e_{d,j}\}_{j\in\mathbb{Z}^{++}}$ for $H(K_d)$ such that

$$W_d e_{d,j} = \lambda_{d,j} e_{d,j} \qquad \forall\, j \in \mathbb{Z}^{++}.$$

We have

$$\|W_d\|_{\mathrm{Lin}[H(K_d)]} = \|\,\mathrm{App}_d\,\|^2_{\mathrm{Lin}[H(K_d),L_2(I^d)]}$$

and

$$\|W\|_{\mathrm{Lin}[H(K)]} = \|\,\mathrm{App}\,\|^2_{\mathrm{Lin}[H(K),L_2(I)]} \le \kappa_1,$$

see [14, Lemma 2].

We summarize the results of [14] in the following Lemmas:

Lemma 4.1. *Let κ_1, κ_2 and σ_d be defined by* (1), (4) *and* (5).

1. *There exists $c_d \in [\kappa_2, \kappa_1]$ such that*

$$\| \operatorname{App}_d \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} = \sigma_d(c_d).$$

2. *If $\kappa_2 = 0$, then*

$$\| \operatorname{App}_d \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} = \max_{\mathfrak{u} \in \mathscr{P}_d} \left[\gamma_{d,\mathfrak{u}} \| W \|^{\mathfrak{u}}_{\operatorname{Lin}[H(K)]} \right]^{1/2}.$$

Lemma 4.2. *Let $d \in \mathbb{Z}^{++}$ and $n \in \mathbb{Z}^{+}$.*

1. *Let*

$$A^*_{d,n}(f) = \sum_{j=1}^{n} \langle f, e_{d,j} \rangle_{H(K_d)} e_{d,j} \qquad \forall f \in H(K_d).$$

Then

$$\| \operatorname{App}_d - A^*_{d,n} \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} \le \frac{\sigma_d(\kappa_1)}{\sqrt{n+1}}.$$

2. *There exist points $\mathbf{t}_1, \ldots, \mathbf{t}_n$ and elements $a_1, \ldots, a_n \in H(K_d)$ such that*

$$A_{d,n}(f) = \sum_{j=1}^{n} f(\mathbf{t}_j) a_j \qquad \forall f \in H(K_d),$$

we have

$$\| \operatorname{App}_d - A_{d,n} \|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} \le \frac{\sigma_d(\kappa_1)\sqrt{2}}{n^{1/4}}.$$

We stress that these results are non-constructive for the class Λ^{std}. Constructive error bounds may be found in [15].

5 General results for quasilinear problems

We first define what we mean by a quasilinear problem, and then present a number of results that guarantee tractability of quasilinear problems.

We say that the problem $S = \{S_d\}_{d \in \mathbb{Z}^{++}}$ is *quasilinear* if for all $d \in \mathbb{Z}^{++}$, the operator $S_d : H(K_d) \times Q_d \to G_d$ satisfies two conditions:

1. For any $q \in Q_d$, we have $S_d(\cdot, q) \in \operatorname{Lin}[H(K_d), G_d]$.

2. There exists a function $\phi \colon H(K_d) \to Q_d$, and a non-negative number C_d, such that

$$\|S_d(f,q) - S_d(\tilde{f}, \phi(\tilde{q}))\|_{G_d} \le C_d \left[\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \tilde{q}\|_{L_2(I^d)} \right]$$
$$\forall [f,q] \in H_{d,\rho_1} \times Q_d,\ [\tilde{f}, \tilde{q}] \in H(K_d) \times H(K_d). \quad (10)$$

We now comment on these conditions. The first condition simply states that S_d is linear if we fix the second argument q. The second condition states that S_d satisfies a Lipschitz condition with respect to both its arguments. We wish to motivate the need for the function ϕ. If we perturb two arguments f and q and obtain $\tilde{f}$ and $\tilde{q}$, then the perturbed $\tilde{f}$ and $\tilde{q}$ are elements of $H(K_d)$. We would like to treat $S_d(\tilde{f}, \tilde{q})$ as a perturbation of $S_d(f, q)$. Unfortunately, $S_d(\tilde{f}, \tilde{q})$ need not be well defined, since the second argument $\tilde{q}$ need not belong to Q_d. However, if we have a function ϕ that maps elements of $H(K_d)$ to the set Q_d, then $S_d(\cdot, \phi(\tilde{q}))$ will be well defined. Going back to our example of the Dirichlet problem, the role of the function ϕ is to guarantee that $\phi(\tilde{q}) \geq 0$.

We now turn to tractability results, which will be derived for the absolute and normalized errors. These errors are linked by the relation (7). We will be able to simultaneously state results for both the absolute and normalized errors by using $\mathrm{ErrCrit}(S_d)$ in the assumptions needed for our estimates. In the remainder of this section, we shall let $\mathrm{card}(\cdot, \cdot, \cdot)$ denote either $\mathrm{card}^{\mathrm{abs}}(\cdot, \cdot, \cdot)$ or $\mathrm{card}^{\mathrm{nor}}(\cdot, \cdot, \cdot)$, as appropriate.

Theorem 5.1. *Let $S = \{S_d\}_{d \in \mathbb{Z}^{++}}$ be a quasilinear problem . Suppose that there exists $\alpha \geq 0$ such that*

$$N_\alpha := \sup_{d \in \mathbb{Z}^{++}} \frac{C_d \| \mathrm{App}_d \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]}}{d^\alpha \, \mathrm{ErrCrit}(S_d)} < \infty. \tag{11}$$

Then

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) \leq 2(\rho_1 + \rho_2)^2 N_\alpha^2 \frac{\sigma_d(\kappa_1)^2}{\| \mathrm{App}_d \|^2_{\mathrm{Lin}[H(K_d), L_2(I^d)]}} \left(\frac{1}{\varepsilon}\right)^2 d^{2\alpha},$$

and

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) \leq \left\lceil 8(\rho_1 + \rho_2)^4 N_\alpha^4 \frac{\sigma_d(\kappa_1)^4}{\| \mathrm{App}_d \|^4_{\mathrm{Lin}[H(K_d), L_2(I^d)]}} \left(\frac{1}{\varepsilon}\right)^4 d^{4\alpha} \right\rceil + 1.$$

Proof. The proof is based on that of [14, Thm. 1]. We first consider the class Λ^{all}. For $n \in \mathbb{Z}^+$, let

$$U^*_{d,n}(f, q) = S_d\left(A^*_{d,\lfloor n/2 \rfloor} f, \phi(A^*_{d,\lfloor n/2 \rfloor} q)\right) \qquad \forall [f, q] \in H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2})$$

where $A^*_{d,\lfloor n/2 \rfloor}$ is as defined in Lemma 4.2. The expression on the right-hand side of this equation is well-defined since $A^*_{d,\lfloor n/2 \rfloor} f \in H(K_d)$. Clearly $U^*_{d,n}$ is an algorithm using at most n evaluations from Λ^{all}. From (10) and Lemma 4.2, we have

$$\begin{aligned}
\|S_d(f, q) - U^*_{d,n}(f, q)\|_{G_d} &\leq C_d \left[\|f - A^*_{d,\lfloor n/2 \rfloor} f\|_{L_2(I^d)} + \|q - A^*_{d,\lfloor n/2 \rfloor} q\|_{L_2(I^d)}\right] \\
&\leq \frac{C_d \sigma_d(\kappa_1)}{\sqrt{\lfloor n/2 \rfloor + 1}} \left[\|f\|_{H(K_d)} + \|q\|_{H(K_d)}\right] \\
&\leq \frac{\sqrt{2}\, C_d(\rho_1 + \rho_2)\sigma_d(\kappa_1)}{\sqrt{n+1}}
\end{aligned}$$

since $\lfloor n/2 \rfloor + 1 \geq (n+1)/2$. This holds for arbitrary $[f, q] \in H_{d,\rho_1} \times (Q_d \cap H_{d,\rho_2})$, and therefore

$$e(A^*_{d,n}, S_d, \Lambda^{\mathrm{all}}) \leq \frac{\sqrt{2}\, C_d(\rho_1 + \rho_2)\sigma_d(\kappa_1)}{\sqrt{n+1}}.$$

Hence

$$\begin{aligned}\operatorname{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) &\le \frac{2\,C_d^2(\rho_1+\rho_2)^2\sigma_d^2(\kappa_1)}{[\mathrm{ErrCrit}(S_d)]^2}\left(\frac{1}{\varepsilon}\right)^2 \\ &= 2(\rho_1+\rho_2)^2\left(\frac{C_d\|\,\mathrm{App}_d\,\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}}{d^{\,\alpha}\,\mathrm{ErrCrit}(S_d)}\right)^2\frac{\sigma_d^2(\kappa_1)}{\|\,\mathrm{App}_d\,\|^2_{\mathrm{Lin}[H(K_d),L_2(I^d)]}}\left(\frac{1}{\varepsilon}\right)^2 d^{\,2\alpha} \\ &\le 2(\rho_1+\rho_2)^2N_\alpha^2\frac{\sigma_d^2(\kappa_1)}{\|\,\mathrm{App}_d\,\|^2_{\mathrm{Lin}[H(K_d),L_2(I^d)]}}\left(\frac{1}{\varepsilon}\right)^2 d^{\,2\alpha},\end{aligned}$$

as claimed.

Now we consider the class Λ^{std}. For $n \ge 2$ let

$$U_{d,n}(f,q) = S_d\left(A_{d,\lfloor n/2\rfloor}f, \phi(A_{d,\lfloor n/2\rfloor}q)\right) \qquad \forall\,[f,q] \in H_{d,\rho_1}\times(Q_d\cap H_{d,\rho_2})$$

where algorithm $A_{d,\lfloor n/2\rfloor}$ is as defined in from Lemma 4.2. The expression on the right-hand side of this equation is well-defined since $A_{d,\lfloor n/2\rfloor}f \in H(K_d)$. Clearly $U_{d,n}$ is an algorithm using at most n evaluations from Λ^{std}. From (10) and Lemma 4.2, we have

$$\begin{aligned}\|S_d(f,q)-U_{d,n}(f,q)\|_{G_d} &\le C_d\left[\|f-A_{d,\lfloor n/2\rfloor}f\|_{L_2(I^d)}+\|q-A_{d,\lfloor n/2\rfloor}q\|_{L_2(I^d)}\right] \\ &\le \frac{\sqrt{2}\,C_d\sigma_d(\kappa_1)}{[(n-1)/2]^{1/4}}\left[\|f\|_{H(K_d)}+\|q\|_{H(K_d)}\right] \\ &\le \frac{2^{3/4}C_d(\rho_1+\rho_2)\sigma_d(\kappa_1)}{(n-1)^{1/4}}\end{aligned}$$

for any $[f,q] \in H_{d,\rho_1}\times(Q_d\cap H_{d,\rho_2})$. This implies that

$$e(A_{d,n}, S_d, \Lambda^{\mathrm{std}}) \le \frac{2^{3/4}C_d(\rho_1+\rho_2)\sigma_d(\kappa_1)}{(n-1)^{1/4}}.$$

Denoting $A = \|\,\mathrm{App}_d\,\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}$, we then obtain

$$\begin{aligned}\operatorname{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) &\le \left\lceil\left(\frac{2^{3/4}C_d(\rho_1+\rho_2)\sigma_d(\kappa_1)}{\varepsilon\ \mathrm{ErrCrit}(S_d)}\right)^4\right\rceil+1 = \left\lceil 8(\rho_1+\rho_2)^4\left(\frac{C_d\sigma_d(\kappa_1)}{\mathrm{ErrCrit}(S_d)}\right)^4\left(\frac{1}{\varepsilon}\right)^4\right\rceil+1 \\ &= \left\lceil 8(\rho_1+\rho_2)^4\left(\frac{C_dA}{d^{\,\alpha}\,\mathrm{ErrCrit}(S_d)}\right)^4\frac{\sigma_d^4(\kappa_1)}{A^4}\left(\frac{1}{\varepsilon}\right)^4 d^{\,4\alpha}\right\rceil+1 \\ &\le \left\lceil 8(\rho_1+\rho_2)^4N_\alpha^4\frac{\sigma_d^4(\kappa_1)}{A^4}\left(\frac{1}{\varepsilon}\right)^4 d^{\,4\alpha}\right\rceil+1,\end{aligned}$$

as claimed. □

Note that the cardinality estimates of Theorem 5.1 consist of several factors:

1. The first factor involves N_α, ρ_1, and ρ_2. This factor is independent of ε and d.
2. The next factor involves $\sigma_d(\kappa_1)$ and $\|\,\mathrm{App}_d\,\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}$. This factor is independent of ε. However at this point, it is unclear whether this factor depends on d.

3. The next factor is a power of $1/\varepsilon$.

4. The last factor is a power of d.

Since we want to use these estimates to establish tractability results, we must resolve the status of the second factor.

We first consider general weights γ, after which we will treat product and finite-order weights.

Theorem 5.2. *Let $S = \{S_d\}_{d\in\mathbb{Z}^{++}}$ be a quasilinear problem . Let κ_1, κ_2, σ_d, and W be as in* (1), (4), (5), *and* (9). *Let α and N_α be as in Theorem* 5.1. *Suppose that there exists $\beta \geq 0$ such that*

$$\Gamma_\beta = \sup_{d\in\mathbb{Z}^{++}} \Gamma_{\beta,d} < \infty,$$

where

$$\Gamma_{\beta,d} = \frac{1}{d^\beta} \frac{\sigma_d^2(\kappa_1)}{\delta_{\kappa_2,0}\left(\max\limits_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \|W\|^{|\mathfrak{u}|}_{\mathrm{Lin}[H(K)]}\right) + (1-\delta_{\kappa_2,0})\sigma_d^2(\kappa_2)},$$

where $\delta_{\kappa_2,0}$ is the Kronecker delta. Then

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) \leq 2(\rho_1+\rho_2)^2 N_\alpha^2 \Gamma_\beta \left(\frac{1}{\varepsilon}\right)^2 d^{2\alpha+\beta}$$

and

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) \leq \left\lceil 8(\rho_1+\rho_2)^4 N_\alpha^4 \Gamma_\beta^2 \left(\frac{1}{\varepsilon}\right)^4 d^{4\alpha+2\beta} \right\rceil + 1.$$

Hence in both classes Λ^{all} and Λ^{std}, the quasilinear problem S is strongly tractable if $\alpha = \beta = 0$ and tractable if $\alpha + \beta > 0$.

Proof. Using Lemma 4.1 and the fact that σ_d is non-increasing, we have

$$\|\operatorname{App}_d\|^2_{\mathrm{Lin}[H(K_d),L_2(I^d)]} \geq \delta_{\kappa_2,0}\left(\max_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \|W\|^{|\mathfrak{u}|}_{\mathrm{Lin}[H(K)]}\right) + (1-\delta_{\kappa_2,0})\sigma_d^2(\kappa_2),$$

from which it follows that

$$\frac{\sigma_d^2(\kappa_1)}{\|\operatorname{App}_d\|^2_{\mathrm{Lin}[H(K_d),L_2(I^d)]}} \leq \Gamma_{\beta,d} d^\beta \leq \Gamma_\beta d^\beta.$$

The desired result now follows from Theorem 5.1. □

Let us see how to apply this result when we have product weights, i.e., when

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j \qquad \forall\, \mathfrak{u} \in \mathscr{P}_d,\, d \in \mathbb{Z}^{++},$$

where $\gamma_1 \geq \gamma_2 \geq \cdots \geq 0$. Similarly to [14, Sect. 3.2], we have the following:

Theorem 5.3. *Consider a quasilinear problem $S = \{S_d\}_{d\in\mathbb{Z}^{++}}$ with product weights. Let κ_1 and κ_2 be as in* (1) *and* (4), *and let α and N_α be as in Theorem* 5.1.

1. *Suppose that*

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

Then $\Gamma_0 < \infty$, so that for both classes Λ^{all} and Λ^{std}, the quasilinear problem S is tractable if $\alpha > 0$, and strongly tractable if $\alpha = 0$. For $\alpha > 0$, we have

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) \le 2(\rho_1 + \rho_2)^2 N_\alpha^2 \Gamma_0 \left(\frac{1}{\varepsilon}\right)^2 d^{\,2\alpha}$$

and

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1 + \rho_2)^4 N_\alpha^4 \Gamma_0^2 \left(\frac{1}{\varepsilon}\right)^4 d^{\,4\alpha} \right\rceil + 1.$$

For $\alpha = 0$, we have

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) \le 2(\rho_1 + \rho_2)^2 N_0^2 \Gamma_0 \left(\frac{1}{\varepsilon}\right)^2$$

and

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1 + \rho_2)^4 N_0^4 \Gamma_0^2 \left(\frac{1}{\varepsilon}\right)^4 \right\rceil + 1.$$

2. *Suppose that*

$$a := \limsup_{d\to\infty} \frac{1}{\ln(d+1)} \sum_{j=1}^{d} \gamma_j < \infty.$$

Then $\Gamma_\beta < \infty$ for $\beta > a(\kappa_1 - \kappa_2)$, and in both classes Λ^{all} and Λ^{std}, the quasilinear problem S is tractable, with

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) \le 2(\rho_1 + \rho_2)^2 N_\alpha^2 \Gamma_\beta \left(\frac{1}{\varepsilon}\right)^2 d^{\,2\alpha+\beta}$$

and

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) \le \left\lceil 8(\rho_1 + \rho_2)^4 N_\alpha^4 \Gamma_\beta^2 \left(\frac{1}{\varepsilon}\right)^4 d^{\,4\alpha+2\beta} \right\rceil + 1.$$

Proof. Since we are using product weights, we have

$$\sigma_d^2(\theta) = \prod_{j=1}^{d} (1 + \theta\gamma_j).$$

We first consider the case where $\sum_{j=1}^{\infty} \gamma_j < \infty$. Then $\sigma_d^2(\theta)$ is uniformly bounded in d. Using Lemma 4.1, we find that for $\kappa_2 > 0$ we have

$$\Gamma_0 = \sup_{d\in\mathbb{Z}^{++}} \Gamma_{0,d} = \sup_{d\in\mathbb{Z}^{++}} \frac{\sigma_d^2(\kappa_1)}{\sigma_d^2(\kappa_2)} \le \prod_{j=1}^{\infty} (1 + \kappa_1\gamma_j) < \infty,$$

whereas for $\kappa_2 = 0$ we have

$$\Gamma_0 = \sup_{d\in\mathbb{Z}^{++}} \Gamma_{0,d} = \sup_{d\in\mathbb{Z}^{++}} \frac{\sigma_d^2(\kappa_1)}{\max_{\mathfrak{u}\in\mathscr{P}_d} \prod_{j=1}^{|\mathfrak{u}|} \gamma_j \|W\|_{\mathrm{Lin}[H(K)]}} < \infty.$$

In this last estimate, we use the fact that $\lim_{j\to\infty}\gamma_j = 0$ implies $\lim_{|\mathfrak{u}|\to\infty}\prod_{j=1}^{|\mathfrak{u}|}\gamma_j\|W\|_{\mathrm{Lin}[H(K)]} = 0$. The rest directly follows from Theorem 5.2.

We now consider the case where a is finite. Choosing $\delta > 0$, there exists an integer d_δ such that

$$\frac{1}{\ln(d+1)}\sum_{j=1}^{d}\gamma_j \le a+\delta \qquad \forall\, d \ge d_\delta.$$

For $d \ge d_\delta$, we then have

$$\sigma_d^2(\theta) = \exp\biggl(\sum_{j=1}^{d}\ln(1+\theta\gamma_j)\biggr) \le \exp\biggl(\theta\sum_{j=1}^{d}\gamma_j\biggr) \le e^{\theta(a+\delta)\ln(d+1)} = (d+1)^{\theta(a+\delta)}.$$

Since $(1+\kappa_1\gamma_j)/(1+\kappa_2\gamma_j) \le 1+(\kappa_2-\kappa_1)\gamma_j$, we have

$$\frac{\sigma_d^2(\kappa_1)}{\sigma_d^2(\kappa_2)} = \prod_{j=1}^{d}\frac{1+\kappa_1\gamma_j}{1+\kappa_2\gamma_j} \le \prod_{j=1}^{d}\bigl(1+(\kappa_1-\kappa_2)\gamma_j\bigr) = \sigma_d^2(\kappa_1-\kappa_2).$$

Now take $\beta = (a+\delta)(\kappa_1-\kappa_2)$. For $\kappa_2 > 0$, we have

$$\Gamma_\beta = \sup_{d\in\mathbb{Z}^{++}}\frac{\sigma_d^2(\kappa_1)}{d^{\,\beta}\sigma_d^2(\kappa_2)} \le \max\left\{\max_{d<d_\delta}\frac{\sigma_d^2(\kappa_1)}{d^{\,\beta}\sigma_d^2(\kappa_2)}, \sup_{d\ge d_\delta}\frac{(d+1)^\beta}{d^{\,\beta}}\right\} < \infty.$$

For $\kappa_2 = 0$ we have $\beta = \kappa_1(a+\delta)$ and

$$\begin{aligned}\Gamma_\beta &= \sup_{d\in\mathbb{Z}^{++}}\frac{\sigma_d^2(\kappa_1)}{d^{\,\beta}\max_{\mathfrak{u}\in\mathscr{P}_d}\prod_{j=1}^{|\mathfrak{u}|}\gamma_j\|W\|_{\mathrm{Lin}[H(K)]}}\\ &\le \max\left\{\max_{d<d_\delta}\frac{\sigma_d^2(\kappa_1)}{d^{\,\beta}\max_{\mathfrak{u}\in\mathscr{P}_d}\prod_{j=1}^{|\mathfrak{u}|}\gamma_j\|W\|_{\mathrm{Lin}[H(K)]}}, \sup_{d\ge d_\delta}\frac{(d+1)^\beta}{d^{\,\beta}\max_{\mathfrak{u}\in\mathscr{P}_d}\prod_{j=1}^{|\mathfrak{u}|}\gamma_j\|W\|_{\mathrm{Lin}[H(K)]}}\right\}\\ &< \infty,\end{aligned}$$

since $\lim_{j\to\infty}\gamma_j = 0$ and $\lim_{|\mathfrak{u}|\to\infty}\prod_{j=1}^{|\mathfrak{u}|}\gamma_j\|W\|_{\mathrm{Lin}[H(K)]} = 0$ as before. Since δ can be arbitrarily small, this proves that $\Gamma_\beta < \infty$ for all $\beta > (\kappa_1-\kappa_2)a$. The rest directly follows from Theorem 5.2. □

We now discuss finite-order weights of order ω, i.e., $\gamma_{d,\mathfrak{u}} \ne 0$ only if $|\mathfrak{u}| \le \omega$ for all $\mathfrak{u} \in \mathscr{P}_d$ and $d \in \mathbb{Z}^{++}$. We need the following lemma.

Lemma 5.1. *Let $d, \omega \in \mathbb{Z}^{++}$.*

1. *Let*

$$P_\omega(d) = \sum_{j=0}^{\omega}\binom{d}{j}.$$

Then

$$P_\omega(d) \le 2\,d^{\,\omega}.$$

2. *Let $\boldsymbol{\gamma}$ be finite-order weights of order ω. Then*

$$\sigma_d(\theta) \le \sqrt{2\,d^{\,\omega}\,\max\{\theta^{\,\omega},1\}\max_{\mathfrak{u}\in\mathscr{P}_d}\gamma_{d,\mathfrak{u}}} \qquad \forall\,\theta\in\mathbb{R}^+.$$

Proof. Let us establish the first part of the Lemma. The case $d = 1$ is straightforward, since $P_\omega(1) = P_1(1)$ for $\omega \geq 1$ and $P_1(1) = 2$. Now suppose that $d \geq 2$. If $\omega = 1$, then

$$P_1(d) = 1 + d \leq 2\,d,$$

whereas if $\omega \geq 2$, we have

$$\begin{aligned} P_\omega(d) &= \sum_{j=0}^{\omega} \frac{d(d-1)\ldots(d-j+1)}{j!} \leq \sum_{j=0}^{\omega} \frac{d^{\,j}}{j!} \leq \left(\frac{1}{d}\sum_{j=0}^{\omega-1} \frac{d^{\,j-\omega+1}}{j!} + \frac{1}{\omega!}\right) d^{\,\omega} \\ &\leq \left(\frac{1}{d}\sum_{j=0}^{\omega-1} \frac{1}{j!} + \frac{1}{\omega!}\right) d^{\,\omega} \leq \left(\frac{e}{d} + \frac{1}{\omega!}\right) d^{\,\omega} \leq (\tfrac{1}{2}e + \tfrac{1}{2}) d^{\,\omega} \leq 2\,d^{\,\omega}. \end{aligned}$$

We now turn to the second part of the Lemma. Let $\gamma_{\max} = \max_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}}$ If $\theta > 1$, then

$$\sigma_d^2(\theta) = \sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}}\theta^{|\mathfrak{u}|} \leq \theta^\omega \sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \leq \theta^\omega \gamma_{\max} \sum_{\substack{\mathfrak{u}\in\mathscr{P}_d \\ |\mathfrak{u}|\leq\omega}} 1 = \theta^\omega \gamma_{\max} P_\omega(d),$$

whereas if $\theta \in [0, 1]$, we have

$$\sigma_d^2(\theta) \leq \sum_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \leq \gamma_{\max} P_\omega(d).$$

Using the first part of the Lemma, we find

$$\sigma_d^2(\theta) \leq \max\{\theta^\omega, 1\}\,\gamma_{\max} P_\omega(d) \leq \max\{\theta^\omega, 1\}\,\gamma_{\max} \cdot 2\,d^{\,\omega},$$

as required. □

We are now ready to apply the results of Theorem 5.1 to the case of finite-order weights.

Theorem 5.4. *Consider a quasilinear problem $S = \{S_d\}_{d\in\mathbb{Z}^{++}}$ with finite-order weights of order ω. Let κ_1 and κ_2 be defined by* (1) *and* (4)*, and N_α by* (11)*.*

1. *Suppose that $\kappa_2 > 0$.*

   (a) *For the class Λ^{all}, we have*

   $$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) \leq 2(\rho_1 + \rho_2)^2 N_\alpha^2 \left(\frac{\kappa_1}{\kappa_2}\right)^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{\,2\alpha}.$$

   (b) *For the class Λ^{std}, we have*

   $$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) \leq \left\lceil 8(\rho_1 + \rho_2)^4 N_\alpha^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{\,4\alpha} \right\rceil + 1.$$

   Hence in both classes Λ^{all} and Λ^{std}, the quasilinear problem S is strongly tractable if $\alpha = 0$, and tractable if $\alpha > 0$.

2. *Suppose that* $\kappa_2 = 0$. *Let*

$$\Gamma = \frac{\max(1, \kappa_1)}{\min(1, \|W\|_{\mathrm{Lin}[H(K)]})}.$$

(a) For the class Λ^{all}, *we have*

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{all}}) \leq 4(\rho_1 + \rho_2)^2 N_\alpha^2 \Gamma^\omega \left(\frac{1}{\varepsilon}\right)^2 d^{\,2\alpha+\omega}.$$

(b) For the class Λ^{std}, *we have*

$$\mathrm{card}(\varepsilon, S_d, \Lambda^{\mathrm{std}}) \leq \left\lceil 32(\rho_1 + \rho_2)^4 N_\alpha^4 \Gamma^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 d^{\,4\alpha+2\omega} \right\rceil + 1.$$

Hence in both classes Λ^{all} *and* Λ^{std}, *the quasilinear problem S is tractable.*

Proof. As in the proof of [14, Thm. 2], we find that if $\kappa_2 > 0$, then the first part of Lemma 4.1 yields

$$\frac{\sigma_d^2(\kappa_1)}{\|\,\mathrm{App}_d\,\|^2_{\mathrm{Lin}[H(K_d),L_2(I^d)]}} = \frac{\sigma_d^2(\kappa_1)}{\sigma_d^2(c_d)} \leq \frac{\sigma_d^2(\kappa_1)}{\sigma_d^2(\kappa_2)} = \frac{\sum_{\mathfrak{u}\in\mathscr{P}_d,\,|\mathfrak{u}|\leq\omega} \kappa_1^{|\mathfrak{u}|}\gamma_{d,\mathfrak{u}}}{\sum_{\mathfrak{u}\in\mathscr{P}_d,\,|\mathfrak{u}|\leq\omega} \kappa_2^{|\mathfrak{u}|}\gamma_{d,\mathfrak{u}}} \leq \left(\frac{\kappa_1}{\kappa_2}\right)^\omega.$$

If $\kappa_2 = 0$, then the second part of Lemma 4.1 yields

$$\begin{aligned}\frac{\sigma_d^2(\kappa_1)}{\|\,\mathrm{App}_d\,\|^2_{\mathrm{Lin}[H(K_d),L_2(I^d)]}} &= \frac{\sigma_d^2(\kappa_1)}{\max_{\mathfrak{u}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} \|W\|^{|\mathfrak{u}|}_{\mathrm{Lin}[H(k)]}} \\ &\leq \frac{\max_{\gamma_{d,\mathfrak{u}}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}}\, \max(1,\kappa_1)^\omega\, P_\omega(d)}{\max_{\gamma_{d,\mathfrak{u}}\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}}\, \min(1, \|W\|_{\mathrm{Lin}[H(K)]})^\omega} = \Gamma^\omega P_\omega(d) \leq 2\,\Gamma^\omega d^{\,\omega}\end{aligned}$$

(we use the first part of Lemma 5.1 in the last step of the second inequality). Using these inequalities in Theorem 5.1, we obtain the desired results. □

As an application of this theorem, we obtain simple conditions that establish strong tractability with finite-order weights.

Theorem 5.5. *Suppose that the hypotheses of Theorem* 5.4 *hold with* $\kappa_2 > 0$. *Furthermore, suppose that either*

$$\rho_3 := \sup_{d\in\mathbb{Z}^{++}} \sum_{u\in\mathscr{P}_d} \gamma_{d,\mathfrak{u}} < \infty \tag{12}$$

and

$$C^* := \sup_{d\in\mathbb{Z}^{++}} \frac{C_d}{\mathrm{ErrCrit}(S_d)} < \infty, \tag{13}$$

or that

$$M := \sup_{d\in\mathbb{Z}^{++}} \frac{\|\,\mathrm{App}_d\,\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]}}{\mathrm{ErrCrit}(S_d)} < \infty \tag{14}$$

and

$$C^{**} := \sup_{d\in\mathbb{Z}^{++}} C_d < \infty. \tag{15}$$

Then the quasilinear problem S is strongly tractable. More precisely:

1. *For the class* Λ^{all}*, we have*

$$\text{card}(\varepsilon, S_d, \Lambda^{\text{all}}) \leq 2(\rho_1 + \rho_2)^2 N_0^2 \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega} \left(\frac{1}{\varepsilon}\right)^2.$$

Hence

$$p_{\text{strong}}(S, \Lambda^{\text{all}}) \leq 2.$$

2. *For* Λ^{std}*, we have*

$$\text{card}(\varepsilon, S_d, \Lambda^{\text{std}}) \leq \left\lceil 8(\rho_1 + \rho_2)^4 N_0^4 \left(\frac{\kappa_1}{\kappa_2}\right)^{2\omega} \left(\frac{1}{\varepsilon}\right)^4 \right\rceil + 1.$$

Hence

$$p_{\text{strong}}(S, \Lambda^{\text{std}}) \leq 4.$$

Here, N_0 *is defined by* (11)*, and satisfies the bound*

$$N_0 \leq \begin{cases} C^* \rho_3^{1/2} \max\{\kappa_1^{\omega/2}, 1\} & \text{if (12) and (13) hold,} \\ C^{**} M & \text{if (14) and (15) hold.} \end{cases} \tag{16}$$

Proof. If (12) and (13) hold, we find that

$$\| \text{App}_d \|_{\text{Lin}[H(K_d), L_2(I^d)]} = \sigma_d(c_d) \leq \sigma_d(\max\{\kappa_1, 1\}) \leq \max\{\kappa_1^{\omega/2}, 1\} \Bigg(\sum_{u \in \mathscr{P}_d} \gamma_{d,\text{u}} \Bigg)^{1/2}$$
$$\leq \rho_3^{1/2} \max\{\kappa_1^{\omega/2}, 1\}.$$

Using this inequality, along with (11), we obtain $N_0 \leq C^* \rho_3^{1/2} \max\{\kappa_2^{\omega/2}, 1\}$. On the other hand, if (14) and (15) hold, we can use (11) to see that $N_0 \leq C^{**} M$. Hence in either case, we find that (16) holds. The remaining results now follow from Theorem 5.4. □

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