What is the complexity of Stieltjes integration?

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Abstract

We study the complexity of approximating the Stieltjes integral $\int_0^1 f(x)\,dg(x)$ for functions f having r continuous derivatives and functions g whose sth derivative has bounded variation. Let $r(n)$ denote the nth minimal error attainable by approximations using at most n evaluations of f and g, and let $\mathrm{comp}(\varepsilon)$ denote the ε-complexity (the minimal cost of computing an ε-approximation). We show that $r(n) \asymp n^{-\min\{r,s+1\}}$ and that $\mathrm{comp}(\varepsilon) \asymp \varepsilon^{-1/\min\{r,s+1\}}$. We also present an algorithm that computes an ε-approximation at nearly-minimal cost.

1 Introduction

Numerical integration is one of the most fundamentally important problems studied by information-based complexity. The IBC literature is replete with hundreds of references to this problem; see the bibliography of Traub and Werschulz (1998), as well as the sources cited therein, for pointers. Most of these papers deal with integrals of the form $\int_D f(x)\,dx$ for a fixed region D and a class of integrands f, or with weighted integrals of the form $\int_D w(x)\,f(x)\,dx$, with a fixed weight function w. For such problems, we have only partial information about the integrands f; this information typically consists of the values of f at a finite set of points. Since the integral is a linear functional of the integrand, the integration problems studied so far have been *linear* problems in the vast majority of cases studied.

In this paper, we look at the complexity of approximating Stieltjes integrals $\int_0^1 f(x)\,dg(x)$. This is a famous classical problem, appearing in the standard texts dealing with integration theory and functional analysis; a particularly well-written discussion may be found in Riesz and Sz-Nagy (1955, pg. 105 ff.). The Stieltjes integral occurs in numerous areas, such as biology (Louie and Somorjai, 1984), chemistry (Cacelli et al., 1988), chemical engineering (Giona and Patierno, 1997), finance (Duffie, 1996), nuclear engineering (Akiba et al., 1996), physics (Avellaneda and Vergassola, 1995), and stochastic differential equations (Revuz and Yor, 1994).

In our study of Stieltjes integration, we shall assume that we have partial information about f and g. This means that we are considering a nonlinear integration problem; more precisely, the problem is bilinear in the sense of Jackowski (1990). It is our belief that most of the linear problems arising in IBC have important nonlinear counterparts; this is only one example.

In this paper, we shall assume that f has r continuous derivatives and that $g^{(s)}$ is of bounded variation. More precisely, we shall assume that f belongs to the unit ball of $C^r([0,1])$ and that $\mathrm{Var}^* g + \mathrm{Var}^* g^{(s)} \le 1$; here Var^* is a slight modification of Var, the usual variation of a function.

The main result of this paper is that $r(n)$, the minimal error attainable if n evaluations of f and g are used, is proportional to $n^{-\min\{r,s+1\}}$. In proving this result, we show how to obtain nth optimal approximations, i.e., approximations U_n, using n evaluations of f and g, whose error is nearly minimal.

Using these minimal error results, we easily find results about the complexity $\mathrm{comp}(\varepsilon)$, i.e., the minimal cost of computing an ε-approximation. First, suppose that $r = 0$. Since the problem is nonconvergent (i.e., there is a cutoff value $\varepsilon_0 > 0$ such that $r(n) \ge \varepsilon_0$ for all n), it then follows that $\mathrm{comp}(\varepsilon) = \infty$ if $\varepsilon > \varepsilon_0$. However, if $r \ge 1$, we find that $\mathrm{comp}(\varepsilon)$ is proportional to $c\,\varepsilon^{-1/\min\{r,s+1\}}$, where c is the cost of a function evaluation; moreover, an approximation U_n, where n is proportional to $\varepsilon^{-1/\min\{r,s+1\}}$, computes an ε-approximation at nearly-minimal cost.

The reader may find the presence of the "$+1$" a little surprising in these results. Indeed, recall that the nth minimal error in computing $\int_0^1 f(x)\,dx$ for f belonging to the unit ball of $C^r([0,1])$ is proportional to n^{-r}, see Bakhvalov (1959). Now the bounded variation of $g^{(s)}$ means that g' is somewhat like an s-times differentiable function. Moreover, for smooth enough g, we have

$$\int_0^1 f(x)\,dg(x) = \int_0^1 f(x)g'(x)\,dx,$$

Hence, the integrand fg' has $\min\{r,s\}$ derivatives, and Bakhvalov's result would lead us to expect the nth minimal error to be proportional to $n^{-\min\{r,s\}}$. However, the nth minimal error for our problem is proportional to $n^{-\min\{r,s+1\}}$.

The "+1" gives us another surprise. Many problems are unsolvable if the problem elements have the minimal of smoothness to make their solution well-defined. For example, Bakhvalov's result says that the problem of integrating continuous functions ($r = 0$) is unsolvable. However, the presence of the "+1" tells us that although minimal smoothness in the choice of f renders the Stieltjes integration problem essentially unsolvable, this problem is solvable if we have only minimal smoothness ($s = 0$) in the choice of g.

We sketch the structure of this paper. In Section 2, we give a precise formulation of the problem to be studied. In Section 3, we prove the lower bound for the problem. The analogous upper bound is proved in Section 4.

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2 Problem formulation

Before describing the problem to be solved, we first recall the definition of Stieltjes integrals; see Riesz and Sz-Nagy (1955, pg. 105 ff.) for further discussion. Let $I = [0,1]$ denote the unit interval. For functions $f, g\colon I \to \mathbb{R}$, consider a partition

$$\Delta = \{x_0, \dots, x_n\} \qquad \text{with} \qquad 0 = x_0 < x_1 < \cdots < x_n = 1, \tag{2.1}$$

with

$$|\Delta| = \max_{0 \le i \le n-1} x_{i+1} - x_i,$$

and a set of points

$$\Xi_\Delta = \{\xi_1, \dots, \xi_n\} \qquad \text{with} \qquad \xi_j \in [x_{j-1}, x_j] \text{ for } 1 \le j \le n.$$

Let

$$\Sigma(f, g; \Delta, \Xi_\Delta) = \sum_{j=1}^{n} f(\xi_j)[g(x_j) - g(x_{j-1})].$$

Then

$$\int_0^1 f(x)\, dg(x) = \lim_{|\Delta| \to 0} \Sigma(f, g; \Delta, \Xi_\Delta)$$

(if this limit exists) is the *Stieltjes integral* of f with respect to g.

It is well-known that the Stieltjes integral exists if f is continuous and g is of *bounded variation*, i.e., if

$$\operatorname{Var} g = \sup_{\Delta} \sum_{j=1}^{n} |g(x_j) - g(x_{j-1})|$$

is finite, with Δ as in (2.1). In what follows, it will be more convenient to use a modified form

$$\operatorname{Var}^* g = \operatorname{Var} \overline{g},$$

of the variation. Here, the correction function $\overline{g}\colon I \to \mathbb{R}$ satisfies

- $\overline{g}(x) = g(x)$ for every point $x \in [0,1]$ at which g is continuous,
- if g is discontinuous at $x \in (0,1)$, then $\overline{g}(x)$ lies between the left-hand limit $g(x-)$ and the right-hand $g(x+)$, and
- $\overline{g}$ is continuous at the endpoints of $[0,1]$.

Then (see DeVore and Lorentz, 1993, pg. 17) $\overline{g}$ is well-defined whenver g is of bounded variation; moreover, $\operatorname{Var}^* g$ is the norm of the linear functional $f \mapsto \int_0^1 f(x)\,dg(x)$ on $C(I)$.

We now describe the problem to be solved, using the standard terminology of information-based complexity, see, e.g., Traub et al. (1988). Let r and s be given nonnegative integers. Our class of problem elements will be $F \times G$, where

$$F = \{\, I \xrightarrow{f} \mathbb{R} : f^{(r)} \text{ is continuous and } \|f\|_{C^r(I)} \le 1 \,\}$$

and

$$G = \{\, I \xrightarrow{g} \mathbb{R} : g^{(s)} \text{ is of bounded variation and } \operatorname{Var}^* g + \operatorname{Var}^* g^{(s)} \le 1 \,\}.$$

Our solution operator $S\colon F \times G \to \mathbb{R}$ is defined as

$$S([f,g]) = \int_0^1 f(x)\,dg(x) \qquad \forall\, [f,g] \in F \times G.$$

Since $\int_0^1 f(x)\,dg(x)$ is well-defined when f is continuous and g is of bounded variation, our solution operator is well-defined.

Let $[f,g] \in F \times G$. We compute an approximation to $S([f,g])$ by first evaluating information about f and g, and then using this information in an algorithm. For our problem, we will compute *standard information*

$$N([f,g]) = [\, f(x_1), \ldots, f(x_m), g(t_1), \ldots, g(t_{n-m}) \,] \tag{2.2}$$

about $[f,g]$. By Jackowski (1990, Theorem 3.2.4), there is essentially no loss of generality in assuming that the information is *nonadaptive*, i.e., the points

$$0 \le x_1 < x_2 < \cdots < x_m \le 1 \qquad \text{and} \qquad 0 \le t_1 < t_2 < \cdots < t_{n-m} \le 1,$$

are independent of f and g.

We obtain an approximation $U([f,g])$ to the solution $S([f,g])$ in the form $U([f,g]) = \phi(N([f,g]))$. Here, ϕ is an *algorithm* using the information N, i.e., a mapping $\phi\colon \mathbb{R}^n \to \mathbb{R}$. We measure the quality of an approximation U by its worst case *error*

$$e(U) = \sup_{[f,g]\in F\times G} |S([f,g]) - U([f,g])|.$$

The nth *minimal radius of information*, defined as

$$r(n) = \inf\{\, e(U) : U \text{ uses information of the form (2.2)} \,\},$$

gives us a benchmark by which we can measure how close our approximation is to being optimal.

The cost of computing $U([f,g])$ is defined as $\mathrm{cost}(U([f,g]))$, which is the weighted sum of the total number of function values of f and g, as well as the number of arithmetic operations and comparisons needed to obtain $U([f,g])$. More precisely, we assume that each evaluation of f or g has cost c and that each arithmetic operation or comparison has unit cost. Then

$$\mathrm{cost}(U) = \sup_{[f,g]\in F\times G} \mathrm{cost}(U([f,g])) \tag{2.3}$$

is the worst case *cost* of U.

Finally, the ε-*complexity* is the minimal cost of computing an ε-approximation, i.e.,

$$\mathrm{comp}(\varepsilon) = \inf\{\, \mathrm{cost}(U) : \text{approximations } U \text{ such that } e(U) \le \varepsilon \,\}.$$

3 A lower bound

In this section, we establish a lower bound on the nth minimal radius for our problem.[1]

Theorem 3.1. *The nth minimal radius has a lower bound*

$$r(n) \succcurlyeq \left(\frac{1}{n}\right)^{\min\{r,s+1\}}.$$

[1] We use $\preccurlyeq$, $\succcurlyeq$, and $\asymp$ in this paper to respectively denote O-, Ω-, and Θ-relations.

Proof. We first claim that

$$r(n) \succcurlyeq \left(\frac{1}{n}\right)^r. \tag{3.1}$$

Indeed, recall the well-known result that the nth minimal radius for integration over the unit ball of $C^r([0,1])$ problem is proportional to n^{-r}; see Bakhvalov (1959) for the original proof, or, e.g., Bakhvalov (1977, pp. 301–304) for a proof in English. Suppose that we choose $g(x) \equiv x$; we then have

$$\int_0^1 f(x)\, dg(x) = \int_0^1 f(x)\, dx,$$

It then follows that the nth minimal radius for the classical integration problem over the unit ball of $C^r([0,1])$ is a lower bound for the nth minimal radius for our problem of Stieltjes integration. Hence (3.1) holds, as claimed.

We now claim that

$$r(n) \succcurlyeq \left(\frac{1}{n}\right)^{s+1}. \tag{3.2}$$

Indeed, choose $f(x) \equiv x$. Integrating by parts, we find that

$$\int_0^1 f(x)\, dg(x) = g(1) - \int_0^1 g(x)\, dx \qquad \forall g \in G.$$

Moreover, it is easy to see that if $g \in C^{s+1}(I)$, then $\mathrm{Var}^* g + \mathrm{Var}^* g^{(s)} \leq \|g\|_{C^{s+1}(I)}$. It then follows that the $(n+1)$st minimal radius for integration over the unit ball in the space $C^{s+1}(I)$ is a lower bound on the nth minimal radius for our problem; here the "$+1''$ is needed to insure that we evaluate $g(1)$, in addition to the other evaluations of g. Once again using the classical result of Bakhvalov, it immediately follows that (3.2) holds, as claimed, which completes the proof of the theorem. □

4 An upper bound

In this section, we exhibit an algorithm having nearly-optimal error. The case $r = 0$ is trivial. Indeed, when $r = 0$, the nth minimal radius does not converge to zero with n, and so the zero algorithm is optimal. In what follows, we shall consider the case of $r \geq 1$.

Choosing

$$k = \max\{r-1, s\}, \tag{4.1}$$

we let P_k denote the space of polynomials of degree at most k. For any positive integer ℓ, we let

$$\Delta = \{t_0, \dots, t_{\ell+1}\}$$

be a uniform partition of I, i.e.,

$$t_i = \frac{i}{\ell+1} \qquad (0 \le i \le \ell+1).$$

Let

$$\mathscr{S}_\Delta = \left\{ v \in C(I) : v\big|_{[t_i, t_{i+1}]} \in P_k \text{ for } 0 \le i \le \ell-1 \right\}$$

denote a spline space of dimension

$$n_\Delta = \dim \mathscr{S}_\Delta = k(\ell+1)+1.$$

We can choose a basis $\{s_1, \dots, s_{n_\Delta}\}$ for $\mathscr{S}_\Delta$ by the condition

$$s_i \in \mathscr{S}_\Delta \text{ satisfies } s_i(x_j) = \delta_{i,j} \qquad (1 \le i, j \le n_\Delta), \tag{4.2}$$

where

$$x_j = \frac{j-1}{n_\Delta - 1} \qquad (1 \le j \le n_\Delta).$$

For any continuous function $v\colon I \to \mathbb{R}$, we let

$$(\Pi_\Delta v)(x) = \sum_{j=1}^{n_\Delta} v(x_j) s_j(x) \tag{4.3}$$

denote the $\mathscr{S}_\Delta$-*interpolant* of v. For $[f,g] \in F \times G$, we let

$$n = 2n_\Delta$$

and define

$$U_n([f,g]) = \int_0^1 (\Pi_\Delta f)(x)(\Pi_\Delta g)'(x)\,dx \tag{4.4}$$

as our approximation to $S([f,g])$.

Note that if we define information N_n as

$$N_n([f,g]) = [f(x_1), \dots, f(x_{n_\Delta}), g(x_1), \dots, g(x_{n_\Delta})] \tag{4.5}$$

then for any $[f,g] \in F \times G$, we have $U_n([f,g]) = \phi_n(N_n([f,g]))$, where

$$\phi_n(N_n([f,g])) = \sum_{i=1}^{n_\Delta} f(x_i) \sum_{j \in \operatorname{supp}(i,\Delta)} \alpha_{i,j} g(x_j), \tag{4.6}$$

with

$$\alpha_{i,j} = \int_0^1 s_i(x)s_j'(x)\,dx \qquad (1 \le i, j \le n_\Delta),$$

and

$$\mathrm{supp}(i,\Delta) = \{\, j \in \{1,\ldots,n_\Delta\} : \mathrm{support}(s_i) \cap \mathrm{support}(s_j) \ne \emptyset \,\}.$$

Hence the approximation U_n is given by an algorithm ϕ_n using information N_n involving n evaluations of f and g.

Let us estimate $\mathrm{cost}(U_n)$. First of all, note that since the coefficients $\alpha_{i,j}$ are independent of $[f,g] \in F \times G$, they may be precomputed. Moreover, since our basis functions $s_1,\ldots s_{n_\Delta}$ satisfy (4.2), we have

$$\kappa = \sup_{\Delta} \max_{1\le i\le n_\Delta} \mathrm{supp}(i,\Delta) < \infty.$$

We then have

Lemma 4.1.

$$\mathrm{cost}(U_n) \le (c + \kappa + \tfrac{1}{2})n - 1.$$

Proof. The first step in calculating $U_n([f,g])$ is the evaluation of the information $N_n([f,g])$ described in (4.5). Hence the information cost for U_n is

$$\mathrm{cost}(N_n[f,g]) = c\,n.$$

Now each inner sum in (4.6) is a sum over at most κ terms, which means that it can be evaluated using $2\kappa - 1$ arithmetic operations. Thus the combinatory cost for calculating each summand for the outer sum is at most 2κ, so that the cost of calculating all the summands for the outer sum is at most $2\kappa n_\Delta$. Finally, there are n_Δ such summands to add, which has an additional cost of $n_\Delta - 1$ arithmetic operations. Hence we see that the combinatory cost for U_n is

$$\mathrm{cost}\left(\phi_n(N_n([f,g]))\right) \le (2\kappa+1)n_\Delta - 1 = (\kappa + \tfrac{1}{2})n - 1.$$

The result now follows from (2.3). ☐

We are now ready to state an error bound for U_n:

Theorem 4.1. *Let $r \ge 1$. Then*

$$e(U_n) \preccurlyeq \left(\frac{1}{n}\right)^{\min\{r,s+1\}}.$$

Before proving Theorem 4.1, we first establish

Lemma 4.2. *If $g^{(s)}$ is of bounded variation, then*

$$\|g-\Pi_\Delta g\|_{L_1(I)} \preccurlyeq \left(\frac{1}{n}\right)^{s+1} \operatorname{Var}^* g^{(s)}.$$

Proof. Since $k \geq s$, we may use Werschulz (1991, Lemma 5.4.3) to see that there exists $C>0$, independent of n, such that for any $w \in W^{s+1,1}(I)$, the inequality

$$\|w-\Pi_\Delta w\|_{L_1(I)} \leq Cn^{-(s+1)}\|w^{(s+1)}\|_{L_1(I)}$$

holds. From DeVore and Lorentz (1993, Chapter 7, Theorem 5.2), it then follows that

$$\|g-\Pi_\Delta g\|_{L_1(I)} \preccurlyeq \omega_{s+1}(g,n^{-1})_1, \tag{4.7}$$

the right-hand side denoting the usual modulus of smoothness in $L_1(I)$, see DeVore and Lorentz (1993, Chapter 2, Definition 7.2). Using DeVore and Lorentz (1993, Chapter 2, formula (7.13)), we have

$$\begin{aligned}\omega_{s+1}(g,n^{-1})_1 &\leq n^{-s}\omega_1(g^{(s)},n^{-1})_1 \\ &= n^{-(s+1)}\cdot n\omega_1(g^{(s)},n^{-1})_1 \\ &\leq n^{-(s+1)}\|g^{(s)}\|_{\mathrm{Lip}(1,L_1(I))}.\end{aligned} \tag{4.8}$$

Here, $\mathrm{Lip}(1,L_1(I))$ is the class of functions v for which

$$\|v\|_{\mathrm{Lip}(1,L_1(I))} = \sup_{t>0} t^{-1}\omega_1(v,t)_1$$

is finite, see DeVore and Lorentz (1993, pg. 5). But DeVore and Lorentz (1993, Chapter 2, Lemma 9.2) implies that

$$\|g^{(s)}\|_{\mathrm{Lip}(1,L_1(I))} \leq \operatorname{Var}^* g^{(s)}. \tag{4.9}$$

The lemma now follows from (4.7), (4.8), and (4.9). ☐

We are now ready to give the

Proof of Theorem 4.1. Let $([f,g]) \in F \times G$. We then have

$$|S([f,g])-U([f,g])| \leq |I_1|+|I_2|, \tag{4.10}$$

where

$$I_1 = \int_0^1 [f(x)-(\Pi_\Delta f)(x)]\,dg(x)$$

and

$$I_2 = \int_0^1 (\Pi_\Delta f)(x)\, d[g(x) - (\Pi_\Delta g)(x)].$$

Clearly

$$|I_1| \le \| f - \Pi_\Delta f\|_{C(I)} \operatorname{Var}^* g.$$

Since $k \ge r-1$, we may use Oden and Reddy (1976, Theorem 6.21) to see that

$$\| f - \Pi_\Delta f\|_{C(I)} \preccurlyeq n^{-r} \| f^{(r)}\|_{C(I)}.$$

Hence

$$|I_1| \preccurlyeq n^{-r} \| f^{(r)}\|_{C(I)} \operatorname{Var}^* g \preccurlyeq n^{-r}. \tag{4.11}$$

Moreover, g and $\Pi_\Delta g$ agree at the endpoints of I, and so an integration by parts yields

$$I_2 = \int_0^1 [g(x) - (\Pi_\Delta g)(x)]\, d(\Pi_\Delta f).$$

Hence

$$|I_2| \le \|g - \Pi_\Delta g\|_{L_1(I)} \|(\Pi_\Delta f)'\|_{L_\infty(I)}.$$

From Lemma 4.2, we have

$$\|g - \Pi_\Delta g\|_{L_1(I)} \preccurlyeq n^{-(s+1)}.$$

A second application of Werschulz (1991, Lemma 5.4.3) yields

$$\|(f - \Pi_\Delta f)'\|_{L_\infty(I)} \preccurlyeq \| f'\|_{L_\infty(I)},$$

so that

$$\|(\Pi_\Delta f)'\|_{L_\infty(I)} \preccurlyeq \| f'\|_{L_\infty(I)} \le 1.$$

Thus

$$|I_2| \preccurlyeq n^{-(s+1)}. \tag{4.12}$$

The result now follows from (4.10), (4.11), and (4.12). □

Combining the results of Theorems 3.1 and 4.1, and using Lemma 4.1, we easily have

Corollary 4.1. *The following results hold for the Stieltjes integration problem:*

1. *Let $r = 0$. Then the problem is nonconvergent. That is, there exists $\varepsilon_0 > 0$ such that*

$$r(n) \geq \varepsilon_0 \qquad \forall n \in \mathbb{N},$$

and thus

$$\mathrm{comp}(\varepsilon) = \infty \qquad \forall \varepsilon < \varepsilon_0.$$

2. *Let $r \geq 1$.*

   (a) *The nth minimal error satisfies*

   $$r(n) \asymp \left(\frac{1}{n}\right)^{\min\{r,s+1\}}.$$

   Moreover, if $k \geq \min\{r-1, s\}$, then the approximation U_n given by (4.4) *satisfies*

   $$e(U_n) \asymp \left(\frac{1}{n}\right)^{\min\{r,s+1\}},$$

   and thus U_n is an nth optimal approximation.

   (b) *The ε-complexity of the problem is*

   $$\mathrm{comp}(\varepsilon) \asymp c \left(\frac{1}{\varepsilon}\right)^{\min\{r,s+1\}},$$

   where c is the cost of a function evaluation. Moreover, we can compute an ε-approximation at nearly-minimal cost by using the approximation U_n, with

   $$n \asymp \left(\frac{1}{\varepsilon}\right)^{\min\{r,s+1\}}.$$

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