Merging Globally Rigid Formations of Mobile Autonomous Agents

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Abstract

This paper is concerned with merging globally rigid formations of mobile autonomous agents. A key element in all future multi-agent systems will be the role of sensor and communication networks as an integral part of coordination. Network topologies are critically important for autonomous systems involving mobile underwater, ground and air vehicles and for sensor networks. This paper focuses on developing techniques and strategies for the analysis and design of sensor and network topologies required to merge globally rigid formations for cooperative tasks. Central to the development of these techniques and strategies will be the use of tools from rigidity theory, and graph theory.

1. Introduction

This paper addresses merging "globally rigid formations." A *formation* is defined as a group of mobile agents moving in real 2- or 3-dimensional space. A formation is *rigid* if the distance between each pair of agents does not change over time, at least under ideal conditions. A formation is called *globally rigid*, if the distance between each pair of agents is unambiguous. Sensing and communication links are used for maintaining fixed distances between agents. It is not necessary to have sensing and communication links between each pair of agents to maintain a rigid formation [2]. Distances between all agent pairs can be held fixed by directly measuring distances between only some agents and keeping them at desired values. It is also true that it is not necessary to have sensing and communication links between each pair of agents to create a globally rigid formation [7]. In [2, 3, 6] Eren et al. introduced approaches based on rigidity and global rigidity for maintaining formations of autonomous agents with sensor and network topologies that use distance, direction, bearing and angle information between agents.

In the context of this paper, "agents" are considered to be autonomous vehicles, robots or sensors such as autonomous underwater vehicles (AUVs), microsatellites, uninhabited air vehicles (UAVs), mobile ground-based robots, and mobile sensors.

A key element in all future multi-agent systems will be the role of sensor and communications networks as an integral part of coordination. In a rigid formation, distances between agents are held fixed by measurements and information gathered through "sensing and communication links" between agents. One of the challenges in building sensor and communications networks between agents is the "topology" of the network. By *topology*, we mean the interconnection structure of sensing and communication links among agents. In other words, topology refers to the network's layout. A network's topology determines how different agents in the network are connected to each other. Two networks have the same topology if the interconnection structure is the same, although the networks may differ in physical interconnections, distances between agents, transmission rates, and signal types. Network topologies are critically important for autonomous systems involving mobile underwater, ground and air vehicles, and for sensor networks. Energy efficiency and communication bandwidth are critically important in formations of mobile autonomous agents, and hence strategies that make efficient use of power and energy are beneficial. Therefore, we use topologies for providing sensing and communications with the minimum number of links, and propose methods requiring the minimum number of changes in the set of links in merging rigid sub-formations. Rigid formations with the minimum number of sensing and communication links required to achieve rigidity are called *minimally rigid formations*.

Formations of autonomous agents usually operate under time-varying conditions where sensor and network topologies need to be restructured. Such conditions can be changes in the environment, obstacles along the trajectories of agents or departures of agents from formation. Eren et al. addressed "operations" on rigid formations in [4, 1]. By an *operation*, we mean missions and maneuvers that include agent departures, splitting, and merging, which result in changes in agent set and/or interconnection structure of sensing and communication links. These operations included maintaining rigidity after an agent departs from a formation, splitting formations, and merging sub-formations. Eren et al. addressed the use of global rigidity in formations of mobile autonomous agents and in network localization problem in [7, 5]. In this paper the approach is extended to the case in which we consider the problem of merging globally rigid formations. By *merging*, we mean two types of operations. The first type is inserting links between globally rigid sub-formations which results in a single post-merged globally rigid formation. The second type is sharing agents between two sub-formations so that the resulting formation is globally rigid . During a merging operation, it is a natural starting point to preserve the links in each pre-merged rigid sub-formation. Hence a reasonable goal is to create a new post-merged rigid formation by inserting a minimum number of links between sub-formations in the first type of merging operation; and to share minimum number of agents between sub-formations in the second type of merging operations. A merging operation, for example, can be used to create one single rigid formation after split sub-formations pass around an obstacle. As a further application of splitting and merging operations, one can consider using both of these operations together when there is a change in a mission. For example, some changes in sensor and network topologies can be achieved by a series of splitting-merging operations by splitting a formation into two or more sub-formations and then merging these sub-formations into one post-merged formation which has a completely different topology of sensing and communication links.

To motivate our discussion of merging a rigid formation, we have the following example:

Example: Consider two globally rigid formations in 2-dimensional space as shown in Figure 1. We would like to merge these two formations resulting in a single rigid formation in such a way that all pairs of links in each formation are preserved and a minimum number of links is inserted between these two formations.

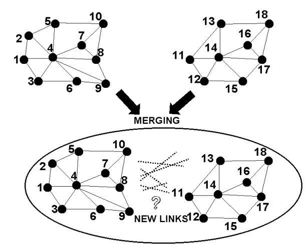

Figure 1. Two formations are merged to form one single globally rigid formation. Finding the new links to be inserted between these two formations, which will make the whole formation rigid, is the merging problem.

The paper is organized as follows: In §2, we review strategies for creating sensor and network topologies of rigid formations with distance information between agents in 2- and 3-dimensional space [2]. In §3, we review strategies for creating sensor and network topologies of globally rigid formations with distance information between agents in 2- and 3-dimensional space [2]. In §4, we present the main results of the paper: strategies to merge globally rigid formations. We end the paper with summary and concluding remarks in §5.

2. Rigid Formations

We start with an overview of rigidity. Recall that a formation is rigid if the distance between each pair of agents does not change over time under ideal conditions. In this section, essentially complete theory of rigid formations in 2-dimensional space is reviewed, as well as known partial results for 3-dimensional space. We review "generic" rigidity, which is the type of rigidity most useful for our purposes. In practice, actual agent groups cannot be expected to move exactly in rigid formation because of sensing, modeling, and actuation errors. With generic rigidity, the topology will be robust for maintaining formations under small perturbations. Although there is no existing comparable complete theory for 3-dimensional space, there are useful partial results [14, 15]. We review sequential techniques to generate rigid classes of formations both in 2- and 3-dimensional space. The approach presented in this section forms the basis of the techniques developed in the subsequent sections.

2.1. Point Formations and Rigidity

By a d-dimensional *point formation* at $p \triangleq \text{column } \{p_1, p_2, \ldots, p_n\}$, written $\mathbb{F}_p$, is meant a set of n points $\{p_1, p_2, \ldots, p_n\}$ in $\mathbb{R}^d$ together with a set $\mathcal{L}$ of k *maintenance links*, labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$; the *length* of link (i, j) is the Euclidean distance between point p_i and p_j. The idea of a point formation is essentially the same as the concept of a "framework" studied in mathematics [14, 12] as well as within the theory of structures in mechanical and civil engineering. For our purposes, a point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ provides a natural high-level model for a set of n agents moving in real 2- or 3- dimensional space. In this context, the points p_i represent the positions of agents in $\mathbb{R}^d$ $\{d = 2 \text{ or } 3\}$ and the links in $\mathcal{L}$ label those specific agent pairs whose inter-agent distances are to be maintained over time. In practice actual agent positions cannot be expected to move exactly in formation because of sensing errors, vehicle modelling errors, etc. The ideal benchmark formation against which the performance of an actual agent formation is to be measured is called a *reference formation.*

Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G} \triangleq (\mathcal{V}, \mathcal{L})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$ and edge set $\mathcal{L}$, as well as a distance function $\delta : \mathcal{L} \to \mathbb{R}$ whose value at $(i, j) \in \mathcal{L}$ is the distance between p_i and p_j. Let us note that the distance function of $\mathbb{F}_p$ is the same as the distance function of any point formation $\mathbb{F}_q$ with the same graph as $\mathbb{F}_p$ provided q is *congruent* to p in the sense that there is a distance preserving map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(q_i) = p_i, i \in \{1, 2, \ldots, n\}$. In the sequel we will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are *congruent* if they have the same graph and if q and p are congruent.

By a *trajectory* of $\mathbb{F}_p$, we mean a continuously parameterized, one-parameter family of points $\{q(t) : t \geq 0\}$ in $\mathbb{R}^{nd}$, which contains p. A point formation $\mathbb{F}_p$ is said to be *rigid* if the distance between every pair of its points remains constant along any trajectory on which the lengths of all of its maintenance links in $\mathcal{L}$ are kept fixed. Alternatively, we can define a rigid point formation as follows: A formation is said to undergo *rigid motion* along a trajectory $q([0, \infty)) \triangleq \{\text{column } \{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ if the Euclidean distance between each pair of points $q_i(t)$ and $q_j(t)$ remains constant all along the trajectory. Let us note that $\mathbb{F}_p$ undergoes rigid motion along a trajectory $q([0, \infty))$ just in case each pair of points $q(t_1), q(t_2) \in q([0, \infty))$ are congruent. The set of points $\mathcal{M}_p$ in $\mathbb{R}^{nd}$ which are congruent to p is known to be a smooth manifold. It is clear that any trajectory along which $\mathbb{F}_p$ undergoes rigid motion must lie completely within $\mathcal{M}_p$; conversely any trajectory of $\mathbb{F}_p$ that lies within $\mathcal{M}_p$ is one along which $\mathbb{F}_p$ undergoes rigid motion. A point formation $\mathbb{F}_p$ is said to be *rigid* if rigid motion is the only kind of motion it can undergo along any trajectory on which the lengths of all links in $\mathcal{L}$ remain constant. Thus, if $\mathbb{F}_p$ is rigid, it is possible to "keep formation" by making sure that the lengths of the formation's maintained links do not change as the formation moves.

Whether a given point formation is rigid or not can be studied by examining what happens to the given point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ with m maintenance links, along the trajectory $q([0, \infty)) \triangleq \{\{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ on which the Euclidean distances $d_{ij} \triangleq ||p_i - p_j||$ between pairs of points (p_i, p_j) for which (i, j) is a link are constant. Along such a trajectory

$$(q_i - q_j) \cdot (q_i - q_j) = d_{ij}^2, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \quad (1)$$

We note that the existence of a trajectory is equivalent to the existence of a piecewise analytic path, with all derivatives at the initial point. It is also equivalent to the existence of a sequence of formations on $p(n), n = 1, 2, \ldots$ with the same measurements, and with $\lim_{n\to\infty} p(n)$ converging to p. Assuming a smooth (piecewise analytic) trajectory, we can differentiate to get

$$(q_i - q_j) \cdot (\dot{q}_i - \dot{q}_j) = 0, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \quad (2)$$

Here, $\dot{q}_i$ is the velocity of point i. The m equations can be collected into a single matrix equation

$$R(q)\dot{q} = 0 \quad (3)$$

where $\dot{q} = \text{column } \{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\}$ and $R(q)$ is a specially structured $m \times dn$ matrix called the *rigidity matrix* [11, 14, 15].

Because any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$, is one along which $\mathbb{F}_p$ undergoes rigid motion, (2) automatically holds along any trajectory which lies within $\mathcal{M}_p$. From this, it follows that the tangent space to $\mathcal{M}_p$ at q, written $\mathcal{T}_q$, must be contained in the kernel of $R(q)$. Since p must be on any such trajectory, it must be true that $\mathcal{T}_p \subset$ kernel $R(p)$. If $\dot{q}$ satisfies (3), then it lies in the tangent space. If the affine span of the points $p_1, p_2, \ldots, p_n$ is $\mathbb{R}^n$ (which means that the points $p_1, p_2, \ldots, p_n$ do not lie on any hyperplane in $\mathbb{R}^n$), then $\mathcal{M}_p$ is $n(n + 1)/2$ dimensional since it arises from the $n(n-1)/2$-dimensional manifold of orthogonal transformations of $\mathbb{R}^n$ and the n-dimensional manifold of translations of $\mathbb{R}^n$ [11]. Thus $\mathcal{M}_p$ is 6-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^3$, and 3-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^2$. We have $\text{rank } R(q) = nd-$ dimension kernel $R(q) \leq nd - n(n + 1)/2$.

We have the following theorem [15]:

Theorem 1. *Assume $\mathbb{F}_p$ is a formation with at least d points in d-space $\{d = 2,$ or $3\}$ where $\text{rank } R(p) = \max\{\text{rank } R(x) : x \in \mathbb{R}^d\}$. $\mathbb{F}_p$ is rigid in $\mathbb{R}^d$ if and only if*

$$\text{rank } R(p) = \begin{cases} 2n-3 & \text{if } d = 2, \\ 3n-6 & \text{if } d = 3. \end{cases}$$

This theorem leads to the notion of the "generic" behavior of rigidity. When the rank is less than the maximum, the formation may still be rigid. However, this type of rigidity is unstable. For almost all small changes in the position of p (or in the lengths of the maintenance links), the formation will no longer be rigid. We are interested in "generic rigidity", a property that will hold for all small changes in p.

2.2. Generic Rigidity

In this section, we define a type of rigidity, called "generic rigidity," that is more useful for our purposes. A point formation $\mathbb{F}_p$ is *generically rigid* if it is rigid for almost all choices of p in $\mathbb{R}^{dn}$. It is possible to characterize generic rigidity in terms of the "generic rank" of R where by R's *generic* or maximal rank we mean the largest value of $\text{rank}\{R(q)\}$ as q ranges over all values in $\mathbb{R}^{nd}$. The following theorem is due to Roth [11].

Theorem 2. *A formation $\mathbb{F}_p$ with at least d points in d-space $\{d = 2,$ or $3\}$ is generically rigid if and only if*

$$\text{generic rank } \{R\} = \begin{cases} 2n-3 & \text{if } d = 2, \\ 3n-6 & \text{if } d = 3. \end{cases}$$

To understand this type of rigidity, it is useful to observe that the set of points p that satisfy the condition $\text{rank } R(p) = \max\{\text{rank } R(x) : x \in \mathbb{R}^d\}$ is a dense open subset of $\mathbb{R}^{nd}$ [11]. Generic rigidity is a property of only the set of maintenance links, or the underlying graph. It does not even claim that $\mathbb{F}_p$ itself is rigid but only that almost all nearby points q give rigid formations $\mathbb{F}_q$. The concept of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}_p$ but examines how well the rigidity of formations can be judged by knowing the vertices and their incidences, in other words, by knowing the underlying graph. A point formation $\mathbb{F}_p$ is *strongly generically rigid* if it is generically rigid and if $\text{rank } R(p) = \text{generic rank } \{R\}$. Hence, a strongly generically rigid point formation is rigid and it remains rigid under small perturbations. For this reason, it is a desirable specialization of the concept of a "rigid formation" for our purposes. We have the following theorem for a strongly generically rigid point formation and a generically rigid graph [14]:

Theorem 3. *For a formation $\mathbb{F}_p$ in d-space with at least d points, the following are equivalent:*

1. *the formation's underlying graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in d-dimensional space ($d = 2, 3$);*
2. *for some p,*

$$rank \ \{R(p)\} = \begin{cases} 2n-3 & \text{if } d = 2, \\ 3n-6 & \text{if } d = 3. \end{cases}$$

3. *for almost all p, the formation $\mathbb{F}_p$ is strongly generically rigid.*

As noted above, the concept of generic rigidity does not depend on the precise distances between the points in $\mathbb{F}_p$. For 2-dimensional space, we have a complete combinatorial characterization of generically rigid graphs, which was first proved by Laman in 1970 [10]. In the theorem below, $|\,.\,|$ is used to denote the cardinal number of a set, i.e., the number of elements in a set.

Theorem 4 (Laman [10]). *A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ (where $\mathcal{L} \neq \emptyset$ or $n > 1$) is generically rigid in 2-dimensional space if and only if there is a subset $\mathcal{L}' \subseteq \mathcal{L}$ satisfying the following two conditions: (1) $|\mathcal{L}'| = 2|\mathcal{V}| - 3$, (2) For all $\mathcal{L}'' \subseteq \mathcal{L}', \mathcal{L}'' \neq \emptyset, |\mathcal{L}''| \leq 2|\mathcal{V}(\mathcal{L}'')| - 3$, where $|\mathcal{V}(\mathcal{L}'')|$ is the number of vertices that are end-vertices of the edges in $\mathcal{L}''$.*

There is no comparable complete result for 3-dimensional space, though there are useful partial results [14, 15]. Although we lack a characterization in 3-dimensional space, there are sequential techniques to generate rigid classes of graphs both in 2- and 3-dimensional space based on the vertex addition, edge splitting and vertex splitting operations [12, 13, 14]. We explain these techniques in the sequel, but before that, we discuss minimal rigidity in the next section.

2.3. Minimal Rigidity

A point formation is *minimally rigid* if removing any link makes it non-rigid. There are $2n - 3$ and $3n - 6$ maintenance links in minimally rigid formations in 2- and 3-dimensional space respectively. A graph is called (generically) minimally rigid in d-space if it is rigid and has exactly $dn - \binom{d+1}{2}$ edges (In the sequel, we use the term rigid graph instead of generically rigid graph unless there is a danger of confusion.).

If a point formation is rigid but not minimally rigid, we say that there is *redundancy* in the link set $\mathcal{L}$ and such a formation is called a redundantly rigid point formation. Let us suppose that a link (i, j) is removed from a rigid point formation. If the formation remains rigid then (i, j) is called a *redundant link* in the initial formation (*redundant edge* in the underlying graph). If adding a link (i, j) does not increase the rank of the rigidity matrix, then we call (i, j) an *implicit link* (*implicit edge* in the underlying graph).

2.4. Sequential Techniques

First, we introduce two operations. One operation is the *vertex addition*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we add a new vertex i with d edges between i and d other vertices in $\mathcal{V}$. The other is the *edge splitting*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we remove an edge (j, k) in $\mathcal{L}$ and then we add a new vertex i with $d+1$ edges by inserting two edges (i, j), (i, k) and $d-1$ edges between i and $d-1$ vertices (other than j, k) in $\mathcal{V}$. The resulting graphs after the vertex addition operation and edge splitting operations are also minimally rigid. A more detailed treatment of these operations can be found in Eren et al. [2, 1]. These two operations are used in Henneberg sequences.

Henneberg Sequences: Henneberg sequences are a systematic way of generating minimally rigid graphs based on the vertex addition and edge splitting operations [12]. In d-space, we are given a sequence of graphs: $\mathbb{G}_d, \mathbb{G}_{d+1}, \ldots, \mathbb{G}_{|\mathcal{V}|}$ such that:

1. $\mathbb{G}_d$ is the complete graph on d vertices;
2. $\mathbb{G}_{i+1}$ comes from $\mathbb{G}_i$ by adding a new vertex either by (i) the vertex addition or (ii) the edge splitting operation.

All graphs in the sequence are minimally rigid in d-space. Figure 2 depicts such a Henneberg sequence in 3-dimensional space.

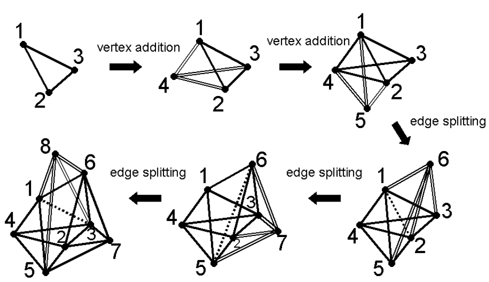

Figure 2. A rigid point formation generated by a Henneberg sequence in 3-dimensional space. Double-lined edges indicate edges created for new vertices. Dashed edges indicate removed edges in the edge splitting operation.

3. Globally Rigid Formations

Global rigidity has been used in formations of mobile autonomous agents and in the network localization problem in the context of sensor networks [5, 7]. The main reason for creating globally rigid formations is that such formations are unambiguous, i.e., the distance between every pair of agents can be determined uniquely. A non-rigid formation has infinitely many "realizations" for the given values of the constraints or dimensions. By a *realization* of a graph $\mathbb{G}$ is meant a function that maps the vertices of $\mathbb{G}$ to points in Euclidean space. Translations, rotations and reflections are not considered to be different realizations. It turns out that even a rigid formation may have several distinct realizations in this sense.

We begin with some notation and vocabulary. Given a formation $(\mathcal{V}; \mathcal{L})$ we have a formation map which takes a configuration p and measures the lengths of edges in $\mathcal{L}$. This can be written as

$$f_{(\mathcal{V};\mathcal{L})} : \mathbb{R}^{d|\mathcal{V}|} \longmapsto \mathbb{R}^{|\mathcal{L}|}.$$

Given a formation $\mathbb{F}_p$, we are interested in what other configurations have the same set of measurements. In other words we are interested in $f^{-1}_{(\mathcal{V};\mathcal{L})}(f_{(\mathcal{V};\mathcal{L})}(p))$. Is this set of configurations a single equivalence class under congruences? In general, as we have seen above, rigidity implies local uniqueness. The converse sometimes fails. Now consider global rigidity for formations in the plane. The appearance of a larger finite number of realizations might come from partial reflections, Figure 3. Generically, we will need vertex 3-connectivity to avoid such reflections if $|V| > 3$. However, it is known that, even if a point formation is rigid and there are no partial reflections, it is still possible to have multiple realizations as shown in Figure 4 ([8]). The following result has recently been announced [9].

Theorem 5. *Given a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ the following are equivalent:*

1. *the graph $\mathbb{G}$ is 3-connected in a vertex sense, and $\mathbb{G}$ is redundantly rigid;*
2. *the formation with distance constraints $(\mathcal{V}, \mathcal{L}, f)$ is globally rigid on generic configurations.*

For 3-space we do know that the following conditions are necessary for generic global rigidity: (i) the graph is 4-connected; (ii) the graph is generically rigid; (iii) the graph is redundantly rigid. These are not sufficient. The counterexample is the graph $K_{5,5}$, the complete bipartite graph on two sets of five vertices.

At the moment we do not have a conjecture for which graphs are generically globally rigid in 3-space. However, we presented the following partial result for subclasses of graphs in Eren et al. [7].

Theorem 6. *A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ with at least $d+2$ vertices ($d = 2, 3$), is generically globally rigid with distance constraints in d-space if there is an ordering of vertices $1, 2, \ldots, |V|$ and a sequence of graphs $\mathbb{G}_{d+2}, \ldots, \mathbb{G}_{|V|}$ such that:*

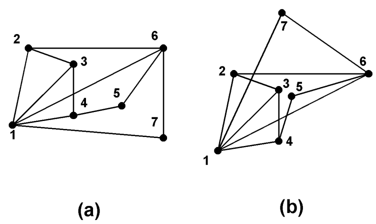

Figure 3. Two realizations with the same weighted graph can be obtained by partial reflections. For example, vertices 5 and 7 are partially reflected in these two realizations.

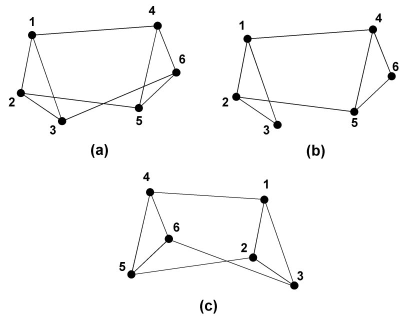

Figure 4. The realizations in a and c have the same underlying distance graph. [8] obtained such two realizations by temporarily removing the edge $(3, 6)$ and rotating the rectangle 1452.

1. *$\mathbb{G}_{d+2}$ is K_{d+2};*

2. *for $d + 2 \leq i \leq |V|$, $\mathbb{G}_{i+1}$ is generated by (i) adding a $d + 1$-valent vertex (ii) edge splitting;*

3. *$\mathbb{G}_{|V|}$ is $\mathbb{G}$.*

4. Results

Now we present the main results of this paper. First, we introduce strategies for merging globally rigid sub-formations by inserting new links between sub-formations. Second, we present the conditions on the number of points shared by sub-formations so that the resulting merged sub-formations are globally rigid.

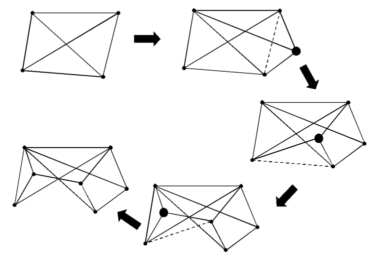

Figure 5. A sequence for generating a globally rigid formation. The sequence starts with K_4, and a new vertex (shown as a larger circle) is adjoined at each step by edge splitting operation. Edges about to be split are shown as dashed lines.

4.1. Connecting Globally Rigid Sub-Formations in 2-Dimensional Space

We have the following theorem:

Theorem 7. *Suppose that two globally rigid sub-formations $\mathbb{F}_1$ and $\mathbb{F}_2$, are connected by a set of links $\mathcal{L}$. Then $\mathbb{F}_1 \cup \mathbb{F}_2 \cup \mathcal{L}$ is globally rigid if the following two conditions hold (see Figure 6):*

1. *The end points of $\mathcal{L}$ has at least three points in $\mathbb{F}_1$ and has at least three points in $\mathbb{F}_2$.*

2. *There are at least four links in $\mathcal{L}$.*

Proof. Let us pick three vertices i, j, k from $\mathbb{F}_2$. By Theorem 6, we apply a sequence of vertex addition and edge splitting operations on i, j, k so that these vertices are connected by four edges to $\mathbb{F}_1$ as shown in Figure 7 and the resulting formation is globally rigid. Now, starting from the three vertices i, j, k, the globally rigid formation $\mathbb{F}_2$ can be created without making any changes in $\mathbb{F}_1$ and the inserted four edges. $\square$

4.2. Connecting Globally Rigid Sub-Formations in 3-Dimensional Space

Theorem 8. *Assume that two globally rigid sub-formations $\mathbb{F}_1$ and $\mathbb{F}_2$, are connected by a set of links $\mathcal{L}$. Then $\mathbb{F}_1 \cup \mathbb{F}_2 \cup \mathcal{L}$ is globally rigid if the following two conditions hold (see Figure 8):*

1. *The end points of $\mathcal{L}$ has at least four points in $\mathbb{F}_1$ and has at least four points in $\mathbb{F}_2$.*

2. *There are at least seven links in $\mathcal{L}$.*

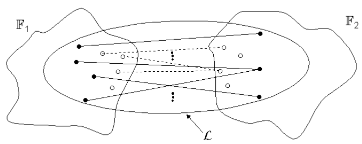

Figure 6. Merging globally rigid sub-formations in 2-dimensional space.

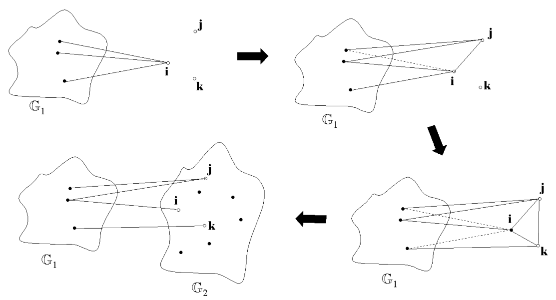

Figure 7. Sequential techniques for creating globally rigid formations are used in the proof of the merging problem in 2-dimensional case. (Please refer to the text.)

Proof. Let us pick three vertices i, j, k, u from $\mathbb{F}_2$. By Theorem 6, we apply a sequence of vertex addition and edge splitting operations on i, j, k, u so that these vertices are connected by seven edges to $\mathbb{F}_1$ as shown in Figure 9 and the resulting formation is globally rigid. Now, starting from the four vertices i, j, k, u, the globally rigid formation $\mathbb{F}_2$ can be created without making any changes in $\mathbb{F}_1$ and the inserted seven edges. □

4.3. Globally Rigid Sub-Formations Sharing Points in 2-Dimensional Space

Theorem 9. *If two globally rigid formations, $\mathbb{F}_1$ and $\mathbb{F}_2$, share at least three points, the formation $\mathbb{F}_1 \cup \mathbb{F}_2$ is globally rigid (Figure 10).*

Proof. First, let us suppose that $\mathbb{F}_1 \cup \mathbb{F}_2$ is not redundantly rigid. Hence there exists an edge (a, b) such that the resulting formation becomes non-rigid if (a, b) is removed, and therefore either $\mathbb{F}_1$ or $\mathbb{F}_2$ becomes also non-rigid. However,

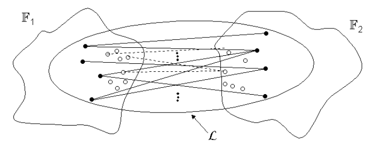

Figure 8. Merging globally rigid sub-formations in 3-dimensional space.

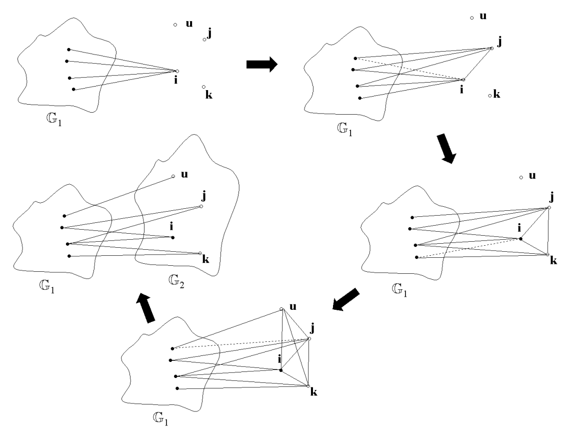

Figure 9. Sequential techniques for creating globally rigid formations are used in the proof of the merging problem in 3-dimensional case. (Please refer to the text.)

this is a contradiction with our initial assumption. Therefore $\mathbb{F}_1 \cup \mathbb{F}_2$ is redundantly rigid. Second, let us consider the 3-connectivity of $\mathbb{F}_1 \cup \mathbb{F}_2$. Given the fact that $\mathbb{F}_1$ and $\mathbb{F}_2$ are 3-connected (since they are globally rigid) and they share three vertices (hence there can be no partial reflections) the resulting graph is also 3-connected. □

4.4. Globally Rigid Sub-Formations Sharing Points in 3-Dimensional Space

Theorem 10. *If two globally rigid formations, $\mathbb{F}_1$ and $\mathbb{F}_2$, share at least four points, the formation $\mathbb{F}_1 \cup \mathbb{F}_2$ is globally rigid (Figure 11).*

Proof. The proof is analogous to the proof in 2-dimensional case. □

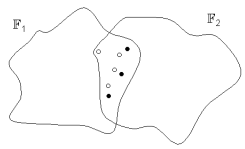

Figure 10. Merging vertex-sharing sub-formations in 2-dimensional space.

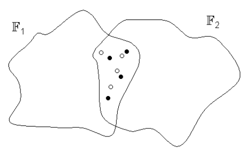

Figure 11. Merging vertex-sharing sub-formations in 3-dimensional space.

5. Summary and Concluding remarks

In this paper, we introduced strategies for merging globally rigid formations of mobile autonomous agents. The results are proved using techniques from rigidity theory. These strategies are applicable in both 2- and 3-dimensional space and for any number of agents. These results will be combined with the techniques for splitting globally rigid formations, which are currently under investigation.

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