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Genus Distributions for Two Classes of Graphs

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The set of orientable imbeddings of a graph can be partitioned according to the genus of the imbedding surfaces. A genus-respecting breakdown of the number of orientable imbeddings is obtained for every graph in each of two infinite classes. It is proved that the genus distribution of any member of either class is strongly unimodal. These are the first two infinite classes of graphs for which such calculations have been achieved, except for a few classes, such as trees and cycles, whose members have all their ~~polygonal~~ orientable imbeddings in the sphere.

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1 Introduction

Gross and Furst [1985] have introduced a hierarchy of genus-respecting partitions of the set of imbeddings of a graph into a closed, oriented surface. This paper contains an illustration of a direct calculation of the genus distribution for every member of an infinite class of graphs called "closed-end ladders". It also contains an illustration of the use of a slightly finer partition in order to obtain the genus distribution for every member of another infinite class of graphs, which are called "cobblestone paths".

The choice of terminology here reflects the usual sensitivities of topological graph theory. For instance, a *graph* may have multiple adjacencies or self-adjacencies. It is taken to be connected, unless one can readily infer otherwise from the immediate context.

We require that the interior of every face of an *imbedding* is simply connected, and we are concerned exclusively with imbeddings into closed, orientable surfaces. The closed orientable surface of genus i is denoted S_i.

Two imbeddings $f : G \to S$ and $g : G \to T$ are called *equivalent* if there exists a homeomorphism of pairs

$$h : (S, f(G)) \to (T, g(G))$$

such that $hf = g$. When we say we are "counting the number of imbeddings," we are actually counting the number of equivalence classes of imbeddings.

The *size of a face* of an imbedding means the number of edge-traversals needed to complete a tour of the face boundary. If both orientations of the same edge appear on the boundary of the same face, then that edge is counted twice in a boundary tour.

It is assumed that the reader is familiar with the elements of topology and graph theory, at the level of White [1984]. However, we shall briefly review the relationship between rotation systems and graph imbeddings, which is described in Section 6.6 of White [1984] in slightly different terminology and somewhat reduced generality.

A *rotation* at a vertex is a cyclic permutation of the edges incident on it, in which the two ends of a self-adjacency are considered separately. Thus, if a vertex has valence d, there are $(d-1)!$ possible rotations there.

A *rotation system* for a graph is an assignment of a rotation to each vertex. If a graph has vertices $V_1, \ldots, V_n$ of respective valences $d_1, \ldots, d_n$, then the total number of rotation systems is

$$\prod_{i=1}^{n} (d_i - 1)!$$

A research abstract of Edmonds [1960] called explicit attention to a bijective correspondence between the set of imbeddings of a graph G and the set of rotation systems. (The correspondence seems to be implicit in the pioneering work of Heffter [1891].) It follows that the total number of imbeddings of a graph is the same as its number of rotation systems. Details for the simplicial case (i.e. without self-adjacencies or multiple adjacencies) were first given by Youngs [1963]. A generalization to the non-simplicial case was developed by Gross and Alpert [1974].

The bijective correspondence is realized if one considers a secondary permutation action of the rotation system. Let e be an oriented edge from vertex u to vertex v. Of course, the primary action takes e onto whatever oriented edge, say d, follows e at vertex v. The secondary action takes e to the reverse of d. The orbits of oriented edges in this secondary action are taken to be the face boundaries of an imbedding.

2 Closed-end ladders

Imagine that rounded pieces of material are used to close both ends of an n-rung ladder. A mathematical model of this object may be obtained by taking the graphical cartesian product of the n-vertex path P_n with the complete graph K_2 and then doubling both its end edges. We call the resulting graph an *n-rung closed-end ladder* and denote it L_n herein. Figure 2.1 depicts a closed-end ladder.

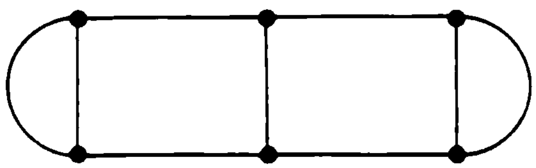

Figure 2.1 The 3-rung closed-end ladder L_3.

The horizontal edges are said to form "sides" of the ladder. The two curved edges are called "ends" or "end-rungs". All vertical edges, including the ones that share vertices with end-rungs, are called "mid-rungs."

Ladder-like graphs played an extensive role in the solution by Ringel and Youngs [1968] to the Heawood Map-Coloring Problem (see Ringel [1974]). In fact, we shall use the picture method of Gustin [1963], so important to that solution, to specify every rotation system — and accordingly, every imbedding — of a ladder graph. We note that a trivalent vertex has only two rotations.

If the vertex is drawn solid, the rotation is counterclockwise. It the vertex is drawn hollow, then the rotation is clockwise. Figure 2.2 shows a rotation system for a 4-rung ladder and its two edge-orbits, one dotted, the other dashed.

The graph L_4 has 8 vertices and 12 edges. The imbedding depicted has two faces (one for each edge-orbit). Substitutions on the right side of the Euler polyhedral equation

$$2 - 2\gamma = V - E + F$$

yields the equation

$$2 - 2\gamma = 8 - 12 + 2 = -2$$

from which we infer that the imbedding surface associated with Figure 2.2 has genus $\gamma = 2$.

If both endpoints of a mid-rung are solid, or if both are hollow, then we call it a *matched mid-rung*. A rung that is either and end-rung or a matched mid-rung is called an *m/e rung*. In particular, an end-rung whose endpoints are unmatched is still called an m/e rung. A rung that is not an m/e rung is called and *unmatched mid-rung*. Thus, Figure 2.2 has three m/e rungs and three unmatched mid-rungs.

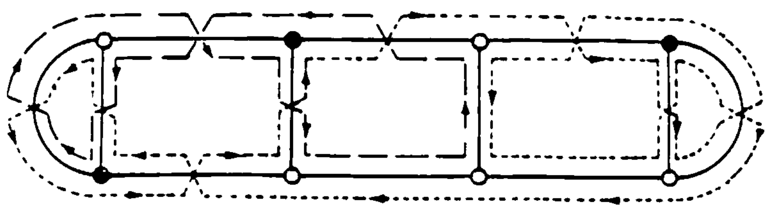

Figure 2.2 A rotation system for the 4-rung ladder L_4 and its two associated edge-orbits.

The m/e rungs are considered to be in a sequence that proceeds from left to right. The left end-rung is first and the right end-rung is last. If the endpoints of an end-rung are matched, then the mid-rung that shares those end-points occurs next to that end-rung in the m/e sequence. Another way to say this is that two m/e rungs are *consecutive m/e rungs* if no matched mid-rungs lie between them.

Two consecutive m/e rungs are said to be *evenly separated* if the number of interposing unmatched rungs is even (including zero). Thus, the left end-rung of Figure 2.2 is evenly separated from the doubly hollow matched rung, but the doubly hollow matched rung is oddly separated from the right end-rung. Thus, the number of edge-orbits (two) is one more than the number of evenly separated pairs (one) of consecutive matched rungs. We generalize this observation about Figure 2.2.

Lemma 2.1 *The number of edge-orbits induced by a rotation system for a closed-end ladder L_n equals one plus the number of evenly separated pairs of consecutive m/e rungs.*

Proof: Suppose that the total number of matched mid-rungs is m. Let us begin by considering any rotation system of the ladder L_m such that every rung is matched, so that there are $m + 1$ evenly separated pairs of m/e rungs. It is not difficult to verify that such a rotation system has $m + 2$ edge-orbits , and that three different edge-orbits are incident on each vertex. (The aid of a few drawings is highly recommended.)

The rest of this proof is concerned with the effect of inserting a string of unmatched mid-rungs between two m/e rungs.

Tracing the orbit lines in Figure 2.3 is sufficient to demonstrate that whenever a 2-string of similar unmatched mid-rungs is inserted between two arbitrary rungs, there is no effect on the number of edge-orbits.

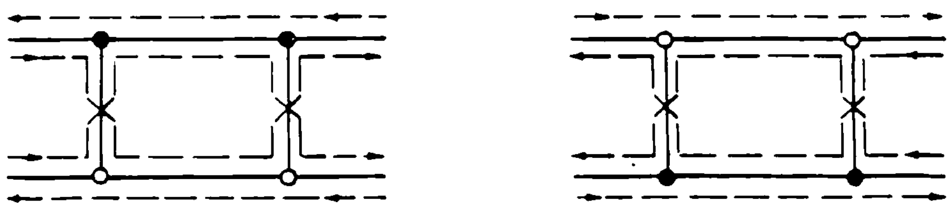

Figure 2.3 The two possible 2-strings of similar unmatched mid-rungs and their local edge-orbit structure.

It follows that when we insert strings of unmatched mid-rungs into the ladder L_m, we may as well assume that consecutive unmatched mid-rungs are dissimilar. Let's call this an *alternating string of unmatched mid-rungs.*

Tracing the edge-orbits in Figure 2.4 indicates that inserting an alternating 3-string of unmatched mid-rungs between any two kinds of rungs has the same effect as inserting only the middle rung of the string.

By combining the observation about alternating 3-strings with the observation about 2-strings of similar m/e rungs, we may infer that the effect of inserting any odd-length string of unmatched mid-rungs is the same as inserting one unmatched mid-rung. Similarly, we may infer that the effect of inserting any even-length string of unmatched mid-rungs is the same as inserting either a 2-string of dissimilar unmatched mid-rungs or no rungs at all.

In order to insert a single unmatched mid-rung between two m/e rungs, we proceed in two stages. First, we insert a m/e rung, which increases the number of edge-orbits by one. We observe that each endpoint of the new rung is incident on three distinct edge-orbits. Thus, when the rotation at one end of the new rung is reversed (i.e. this is stage two), its three edge-orbits become one orbit, for a reduction by two. The net effect of inserting the unmatched mid-rung is a decrease of one edge-orbit.

Another edge-tracing argument confirms that inserting an alternating pair of unmatched mid-rungs between two consecutive m/e rungs causes no net change in the number of edge orbits.

QED

Lemma 2.1 enables us to complete the derivation of the genus distribution of ladders by straightforward enumerative techniques.

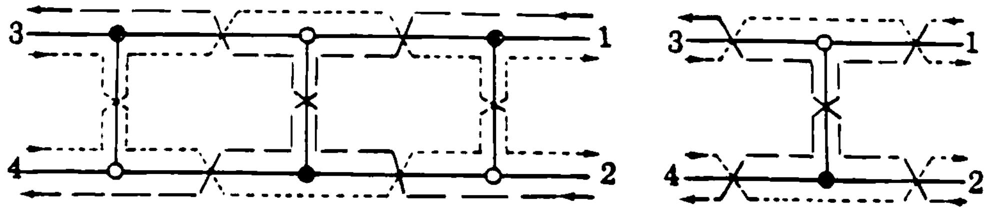

Figure 2.4 The equivalence between inserting an alternating 3-string of unmatched mid-rungs and inserting only the middle rung of the string.

We employ two auxiliary expressions in what follows. One is $s(n, m, k)$, which stands for the number of rotation systems for the ladder L_n that have m m/e mid-rungs, of which k pairs are evenly separated. The other is $b(p, q, r)$, which stands for the number of ways to put p identical balls into q distinct boxes, so that exactly r boxes have an even number of balls.

To obtain a combinatorial expression for $b(p, q, r)$, we imagine that one ball is placed into each of the $q - r$ odd boxes and that the remaining $p - q + r$ balls are then distributed in pairs into the q boxes. Thus,

$$b(p,q,r) = \begin{cases} 0 & \text{if } p-q+r \text{ is odd,} \\ \binom{q}{q-r}\binom{(p-q+r)/2+q-1}{q-1} & \text{otherwise} \end{cases}$$

or, equivalently,

$$b(p,q,r) = \begin{cases} 0 & \text{if } p-q+r \text{ is odd,} \\ \binom{q}{r}\binom{(p-q+r)/2+q-1}{q-1} & \text{otherwise} \end{cases} \quad (1)$$

In order to analyze $s(n, m, k)$ we imagine that the $n - m$ unmatched non-end rungs are to be inserted into the $m + 1$ distinct boxes formed along the ladder L_n by the m unmatched rungs. Clearly we have

$$s(n,m,k) = 2_n b(n-m, m+1, k) \quad (2)$$

If $n \equiv k \bmod 2$, then $(n-m)-(m+1)+k$ is odd, from which it follows that $s(m, n, k) = 0$. However, if $n \not\equiv k \bmod 2$, then we combine equations 1 and 2 to obtain

$$s(n,m,k) = 2^n \binom{m+1}{k}\binom{(n+k-1)/2}{m} \quad (3)$$

We now define $f(n,k)$ to be the number of imbeddings of the ladder graph L_n that have k faces. According to Lemma 2.1 we have

$$f(n,k) = \sum_{m=0}^{n} s(s,m,k-1) \tag{4}$$

Using equation 3, we transform equation 4 into

$$f(n,k) = 2^n \sum_{m=0}^{n} \binom{m+1}{k-1} \binom{(n+k-2)/2}{m} \tag{5}$$

or, equivalently

$$f(n,k) = 2^n \sum_{m=0}^{n} \binom{m+1}{k-1} \binom{(n+k)/2-1}{m} \tag{6}$$

Using the combinatorial identity

$$\binom{m+1}{k-1} = \binom{m}{k-1} + \binom{m}{k-2} \tag{7}$$

we convert equation 6 into

$$f(n,k) = 2^n \sum_{m=0}^{n} \left[\binom{m}{k-1} + \binom{m}{k-2} \right] \binom{(n+k)/2-1}{m} \tag{8}$$

which separates, in turn, into the form

$$f(n,k) = 2^n \sum m = 0^n \binom{m}{k-1} \binom{(n+k)/2-1}{m} + 2^n \sum_{m=0}^{n} \binom{m}{k-2} \binom{(n+k)/2-1}{m} \tag{9}$$

The combinatorial identity

$$\sum_{q=0}^{p} \binom{q}{r} \binom{p}{q} = \binom{p}{r} 2_{p-r}$$

enables us to determine from equation 9 that

$$f(n,k) = 2^n \left[\binom{(n+k)/2-1}{k-1} 2^{(n-k)/2} + \binom{(n+k)/2-1}{k-2} 2^{(n-k)/2+1} \right] \tag{10}$$

Therefore, we may infer

$$f(n,k) = 2^{(3n-k)/2} \left[\binom{(n+k)/2-1}{k-1} + 2 \binom{(n+k)/2-1}{k-2} \right] \tag{11}$$

We now reuse the combinatorial identity 7 to obtaine

$$f(n,k) = 2^{(3n-k)/2} \left[\binom{(n+k)/2}{k-1} + \binom{(n+k)/2-1}{k-2} \right] \tag{12}$$

The combinatorial identity

$$\binom{p-1}{q-1} = \binom{p}{q}\frac{p}{q}$$

implies that

$$\binom{(n+k)/2-1}{k-2} = \binom{(n+k)/2}{k-1}\frac{2(k-1)}{(n+k)}$$

This allows us to simplify equation 12 to conclude

$$f(n,k) = 2^{(3n-k)/2}\binom{(n+k)/2}{k-1}\left(1+\frac{2k-2}{n+k}\right) \tag{13}$$

whenever $n \equiv k \bmod 2$. Otherwise, $f(n,k) = 0$.

In order to convert the face-count formula in equation 13 into a genus distribution formula, we use the Euler polyhedral equation in the form

$$2 - 2i = \#V(L_n) - \#E(L_n) + \#F(L_n \to S_i) = 2n - 3n + k$$

Thus, when the genus of the imbedding surface is equal to the number i, the number of faces is

$$k = n + 2 - 2i$$

Let $g_i(L_n)$ denote the number of imbeddings of the ladder L_n in the surfaces S_i. It follows that

$$g_i(L_n) = f(n, n+2-2i) \tag{14}$$

When we apply equation 13 to the right-side of equation 14, we obtain the equation

$$g_i(L_n) = 2^{[3n-(n+2-2i)]/2}\binom{[n+(n+2-2i)]/2}{(n+2-2i)-1}\left(1+\frac{2(n+2-2i)-2}{n+(n+2-2i)}\right)$$

This simplifies routinely to

$$g_i(L_n) = 2^{n-1+i}\binom{n+1-i}{n+1-2i}\frac{2n+2-3i}{n+1-i}$$

and, since $\binom{a}{a-b} = \binom{a}{b}$, we have

$$g_i(L_n) = \begin{cases} 2_{n-1+i}\binom{n+1-i}{i}\frac{2n+2-3i}{n+1-i} & \text{for } i \le \lfloor\frac{n+1}{2}\rfloor \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

The following table shows the genus distribution for small values of n and i.

	g_0	g_1	g_2	g_3	g_4
L_1	2	2	0	0	0
L_2	4	12	0	0	0
L_3	8	40	16	0	0
L_4	16	112	128	0	0
L_5	32	288	576	128	0

3 Cobblestone paths

Suppose that every edge of the n-vertex path P_n is doubled, and that a self-adjacency is then added at each end. Figure 3.1 shows how the resulting graph might be drawn. It seems appropriate to dub this graph a *cobblestone path* We denote it J_n herein.

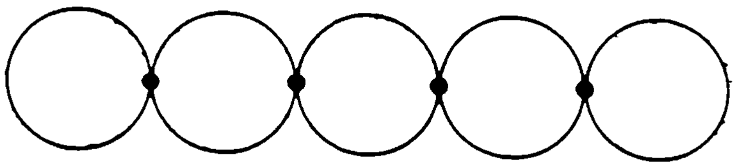

Figure 3.1 The cobblestone path J_4.

For any connected graph G, and for $i = 0, 1, \ldots$, let $g_i(G)$ be the number of imbeddings of G into the closed orientable surface S_i. We may regard the *genus distribution* for G as a vector

$$(g_0(G),\ g_1(G),\ g_2(G), \ldots)$$

Obviously, only finitely many entries are non-zero.

Our objective is to calculate all the numbers $g_i(J_n)$, for $i = 0, 1, \ldots$, and for $n = 1, 2 \ldots$. Sometimes we abbreviate $g_i(J_n)$ by $g_{i,n}$ herein.

The recursion construction assures that we have a cobblestone path J_{n-1} positioned horizontally, as in Figure 3.1. The subsequent cobblestone path J_n is obtained by first imposing a new vertex at the middle of the right end loop and then attaching a new right end loop at the new vertex.

To establish a recursion formula, it is necessary to distinguish between two kinds of imbeddings of the cobblestone path J_{n-1}, depending on whether the two occurrences of the right end edge lie on two distinct faces or on the same face. For $i = 0, 1, \ldots$ and for $n = 1, 2, \ldots$ we define $d_i(J_n)$, sometimes abbreviated $d_{i,n}$, to be the number of imbeddings of J_n in S_i such that the two occurrences of the right-end edge lie on distinct faces, and we define $s_i(J_n)$, sometimes abbreviated $s_{i,n}$, to be the number of imbeddings of J_n in S_i

such that the two occurrences of the right-end edge both lie on the same face. Obviously, we have the equation

$$g_i(J_n) = d_i(J_n) + s_i(J_n) \tag{16}$$

For each cobblestone path Jn, we may form vectors

$$(d_0(J_n),\ d_1(J_n),\ \ldots) \quad \text{and} \quad (s_0(J_n),\ s_1(J_n),\ \ldots).$$

The basis step for the recursion is the following observation

$$(d_{0,1},\ d_{1,1},\ d_{2,1}\ldots) = (4, 0, 0, \ldots) \tag{17}$$

$$(s_{0,1},\ s_{1,1},\ s_{2,1}, \ldots) = (0, 2, 0, \ldots) \tag{18}$$

In constructing the cobblestone path J_n from its predecessor J_{n-1}, we are adding a new vertex of valence 4. Since $(4-1)! = 6$, it follows that J_n has six times as many imbeddings as J_{n-1}. In fact, the cobblestone path J_n has 6^n imbeddings, for $n = 1, 2, \ldots$.

Our viewpoint is that each individual imbedding of J_{n-1} gives rise to six imbeddings of J_n, which occurs by way of the intermediate graph J_{n-1}^{+}. By J_{n-1}^{+} we mean the result of inserting a new vertex at the midpoint of this right-end loop of the cobblestone path J_{n-1}. The six dashed arcs in Figure 3.2 illustrate the six ways the new right-end loop for J_n can be attached at the new vertex of J_{n-1}.

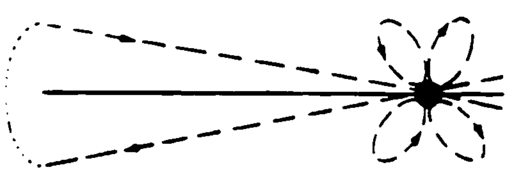

Figure 3.2 The six ways of attaching a new right-end loop.

Now consider any imbedding into the surface S_i of the cobblestone path J_{n-1}. If both occurrences of the right-end loop of J_{n-1} are on the same face, then every one of the six ways of attaching a new right-end loop can be realized in the surface S_i, that is, without attaching an extra handle to S_i. Obviously, the two occurrences of the new right-end loop appear on different faces of the resulting imbedding of J_n. However, if the two occurrences of the right-end loop of J_{n-1} are on different faces of its imbedding in S_i, then only the four monogon-generating insertions of the new loop are in S_i. The two insertions in which the new right-end loops runs from one face to another require the addition of a handle from one face to the other. In this case, both occurrences of the new right-end loop lie on the same face of the new imbedding. Thus, we have established the simultaneous recursion formulae

$$d_i(J_n) = 4d_i(J_{n-1}) + 6s_i(J_{n-1}) \tag{19}$$

$$s_i(J_n) = 2d_{i-1}(J_{n-1}) \tag{20}$$

The solution of the recurrence begins with a substitution of $2d_{i-1}(J_{n-2})$ for $s_i(J_{n-1})$ into equation 20 which yields the simplified recurrence relation

$$d_i(J_n) = 4d_i(J_{n-1}) + 12d_{i-1}(J_{n-2}) \tag{21}$$

By reversing the recursion, we may calculate values

$$d_0(J_0) = 1 \text{ and } d_1(J_0) = d_2(J_0) = \cdots = 0 \tag{22}$$

This artifice enables us to define

$$D_i(x) = \sum_{i=0}^{\infty} d_i(J_n)x^n$$

in preparation for an infinite summation on equation 21, as follows.

$$\sum_{n=2}^{\infty} d_{i,n}x^n = 4\sum_{n=2}^{\infty} d_{i,n-1}x^n + 12\sum_{n=2}^{\infty} d_{i-1,n-2}x^n$$
$$= 4x\sum_{n=2}^{\infty} d_{i,n-1}x^{n-1} + 12x^2\sum_{n=2}^{\infty} d_{i-1,n-2}x^{n-2}$$

Therefore,

$$D_i(x) - d_{i,1}x - d_{i,0} = 4x(D_i(x) - d_{i,0}) + 12x^2 D_{i-1}(x)$$

and, consequently

$$D_i(x)(1-4x) = 12x^2\, D_{i-1}(x) = d_{i,1}\, x + d_{i,0} \tag{23}$$

From equations 17 and 22, we know that $d_{i,1} = 0$ and $d_{i,0} = 0$, for all $i \geq 1$. Thus we may simplify equation 23 to the linear recursion

$$D_i(x) = \frac{12x^2}{(1-4x)} D_{i-1}(x), \quad \text{for } i \geq 1 \tag{24}$$

We will now proceed to establish the value of the polynomial $D_0(x)$. From the Jordan curve theorem, we know that $s_{0,n} = 0$, for $n \geq 1$. Accordingly, we may conclude from equation 19 that

$$d_0(J_n) = 4d_0(J_{n-1})$$

Since $d_{0,0} = 1$, we infer that

$$D_0(x) = \frac{1}{1-4x} \tag{25}$$

We easily combine equations 24 and 25, to obtain the result

$$D_i(x) = \frac{(12x^2)^i}{(1-4x)^{i+1}} \tag{26}$$

The coefficient of x^r in the power series expansion of $(1-ax)^{-s}$ is

$$\binom{r+s-1}{r} a^r$$

(For instance, see Tucker [1980, p. 84] or Liu [1968, p. 31].) It follows that the coefficient of x^{n-2i} in the power series expansion of $(1-4x)^{-(i+1)}$ is

$$\binom{(n-2i)+(i+1)-1}{n-2i} 4^{n-2i} = \binom{n-i}{n-2i} 4^{n-2i}$$

Thus, the coefficient of x^n in the power series expansion of $D_i(x)$ is

$$12^i \cdot 4^{n-2i} \cdot \binom{n-i}{n-2i} = 3^i \cdot 4^{n-i} \cdot \binom{n-i}{n-2i} = 3^i 4^{n-i} \cdot \binom{n-i}{i}$$

That is,

$$d_i(J_n) = 3^i \cdot 4^{n-i} \cdot \binom{n-i}{i}, \text{for} i \geq 0 \text{ and } n \geq 0 \tag{27}$$

We now recall equation 20

$$s_i(J_n) = 2d_{i-1}(J_{n-1}), \text{ for } i \geq 1 \text{ and } n \geq 1$$

and infer that

$$s_i(J_n) = 2 \cdot 3^{i-1} \cdot 4^{n-i} \cdot \binom{n-i}{i-1} \tag{28}$$

Therefore, from equation 16, we conclude

$$g_i(J_n) = 3^i \cdot 4^{n-i} \cdot \binom{n-i}{i} + 2 \cdot 3^{i-1} \cdot 4^{n-i} \cdot \binom{n-i}{i-1} \text{ for } i \geq 0 \text{ and } n \geq 1 \tag{29}$$

The following table contains some of the small values.

	g_0	g_1	g_2	Total
J_1	4	2	0	6
J_2	16	20	0	36
J_3	64	128	24	216
J_4	256	704	336	1296

4 Statistical Patterns

A non-negative sequence $\{k_n\}$ is said to be *unimodal* if there exists at least one integer M such that

$$p_{n-1} \leq p_n \text{ for all } n \leq M, \text{ and}$$
$$p_n \geq p_{n+1} \text{ for all } n \geq M$$

Although this includes non-decreasing sequences that eventually level off and non-increasing sequences that start out level, a typical unimodal sequence first rises and then falls.

A sequence $\{k_n\}$ is called *strongly unimodal* if its convolution with any unimodal sequence is unimodal. Keilson and Gerber [1971] have proved that $\{k_n\}$ is strongly unimodal if and only if

$$k_n^2 \geq k_{n+1}k_n - 1, \text{ for all } n$$

or equivalently, if and only if $\{k_n\}$ is unimodal and

$$\frac{k_{n+1}}{k_n} \leq \frac{k_n}{k_{n-1}}$$

whenever these ratios are defined.

Theorem 4.1 *The genus distribution for closed-end ladders is strongly unimodal.*

Proof: For $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$, we have proved in Section 2 that

$$g_i(L_n) = 2_{n-i+1}\binom{n+1-i}{i}\frac{2n+2-3i}{n+1-i}$$

It follows that the ratio $g_i(L_n)/g_i(L_{n-1})$ has the value

$$2 \cdot \frac{n+3-2i}{i} \cdot \frac{n+2-2i}{n+1-i} \cdot \frac{2n+2-3i}{2n+5-3i}$$

Each of the three quotients is a non-increasing function of the variable i. Thus, the next ratio $g_{i+1}(L_n)/g_i(L_n)$ cannot be greater.

QED

Theorem 4.2 *The genus distribution for a cobblestone path is strongly unimodal.*

Proof: First, transform the equation

$$g_i(J_n) = 3^i 4^{n-i} + 2 \cdot 3^{i-1} 4^{n-i}\binom{n-i}{i-1}$$

so that the right-hand side has only one term.

$$g_i(J_n) = 3^{i-1} 4^{n-i}\binom{n-i}{i}\frac{3n-4i+3}{n-2i+1}$$

Therefore, the ratio $g_i(J_n)/g_{i-1}(J_n)$ equals

$$\frac{3}{4} \cdot \frac{n-2i+2}{n-i+1} \cdot \frac{n-2i+3}{i} \cdot \frac{3n-4i+3}{3n-4i+7}$$

For $1 \le i \le \lfloor \frac{n}{2} \rfloor$, the three variable quotients are non-increasing. Thus, the next ratio $g_{i+1}(J_n)/g_{i-1}(J_n)$ cannot be greater.

QED

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