A Note on Bivariate

Box Splines on a k-direction Mesh

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Abstract.

We determine the dimension of the polynomial subspace of the linear space spanned by the translates over lattice points of a bivariate box spline on a k-direction mesh.

Keywords: Box spline, k-direction mesh.

1. Introduction.

Box splines were introduced by de Boor and De Vore, [1], and systematically studied by de Boor and Hollig in [2,3] and by Dahmen and Micchelli in [4,5,7].

For box splines on a k-direction mesh in an s-dimensional space, one is interested in the dimension of the polynomial subspace of the linear space spanned by the translates of a box spline over lattice points in Z^s, since this dimension is closely related to the rate of approximation using box splines. This problem has been solved for the case $s = 2$ and $k = 4$. For the proof see [4]. For the general case, the result has been announced in a recent paper [6]. The authors suggest to prove it by employing an induction. In this paper, we provide a proof of this result for the case of bivariate box splines (i.e., $s = 2$) and arbitrary k. Our proof does not use induction.

A k-direction mesh is a set of vectors

$$(1.1)\qquad X = \{\underbrace{v^1,\dots,v^1}_{m_1},\dots,\underbrace{v^k,\dots,v^k}_{m_k}\},$$

where $v^i = (\alpha_i,\beta_i) \in \mathbf{Z}^2$, $\mathbf{Z}$ is the set of integers, and $\beta_i \geq 0$, $\alpha_i\beta_j \neq \alpha_j\beta_i$ for $i \neq j$, $m_i \geq 1$, $i,j = 1,\dots,k$, $k \geq 2$. Let

$$(1.2)\qquad n = \Sigma_{i=1}^{k} m_i \quad \text{and} \quad d = \min_i\{n - m_i\} - 1.$$

Then there exists a unique function $B(\cdot|X)$ in R^2, called a box spline on a k-direction mesh, such that

$$(1.3)\qquad \int_{R^2} f(x,y)B((x,y)|X)\,dxdy = \int_0^1\cdots\int_0^1 f(t_1v^1+\ldots+t_nv^k)\times dt_1\ldots dt_n$$

for all $f \in C(R^2)$. The box spline $B(\cdot|X)$ is a piecewise polynomial in $C^{d-1}(R^2)$ where d is given in (1.2), and has compact support [2,3].

Let $S(X)$ be the linear span of translates of the box spline over lattice points in Z^2, i.e.,

$$(1.4)\qquad S(X) = \operatorname{span}(\{B(\cdot-(\alpha,\beta)|X):\ (\alpha,\beta)\in Z^2\}).$$

We are particularly interested in the subspace $S_\pi(X)$ of polynomials in $S(X)$. Let

$$Q_\ell(x,y) = \prod_{i\neq\ell}(\alpha_i x+\beta_i y)^{m_i},$$

and let

$$(1.5)\qquad Q_\ell(D) = \prod_{i\neq\ell}\left(\alpha_i\frac{\partial}{\partial x}+\beta_i\frac{\partial}{\partial y}\right)^{m_i}.$$

It was proved [3,6,7] that $S_\pi(X)$ is of finite dimension and that

$$(1.6)\qquad S_\pi(X) = \mathcal{D}(X),$$

where

$$(1.7)\qquad \mathcal{D}(X) = \{f:\ Q_\ell(D)\,f=0,\ \ell=1,\ldots,k\}.$$

It was proved [3]

<u>Theorem 1.1</u>: If $\det(v^i, v^j) = 1$ for each pair of vectors in X which spans R^2, then

$$(1.8) \qquad \dim S_\pi(X) = \dim \mathcal{D}(X) = A(X),$$

where $A(X)$ is the area of the support of $B(\cdot|X)$. $\square$

A simple example is the case of a 3-direction mesh, where $X = \{\underbrace{e^1,\ldots,e^1}_{m_1}, \underbrace{e^1+e^2,\ldots,e^1+e^2}_{m_2}, \underbrace{e^2,\ldots,e^2}_{m_3}\}$, $e^1 = (1.0)$, $e^2 = (0,1)$. We have

$$\dim_\pi S\ (X) = \dim \mathcal{D}(X) = A(X) = \Sigma_{1\le i<j\le 3}\, m_i m_j.$$

In general, the condition in Theorem 1.1 does not hold, as in the case of a 4-direction mesh, where

$$X = \{\underbrace{e^1,\ldots,e^1}_{m_1}, \underbrace{e^1+e^2,\ldots,e^1+e^2}_{m_2}, \underbrace{e^2,\ldots e^2}_{m_3}, \underbrace{e^2-e^1,\ldots,e^2-e^1}_{m_4}\},$$

since $\det(e^1+e^2, e^2-e^1) = 2$. We address this problem in section 2.

2. Box splines on a k-direction mesh.

For a k-direction mesh as given in (1.1), we give the dimension of $S_\pi(X)$ in Theorem 2.1.

We need the following

<u>Lemma 2.1:</u> Let $(\alpha_i, \beta_i) \in Z^2$ with $\alpha_i\beta_j \neq \alpha_j\beta_i$, $i,j = 1,\ldots,k$, and let $G_j(\lambda) = \Pi_{i\neq j}(\alpha_i+\beta_i\lambda)^{m_i}$, where $m_i \geq 1$. Then for all distinct $\lambda_0,\ldots,\lambda_{n-1}$, the following matrix is nondegenerate:

$$(2.1) \qquad M_n = \begin{pmatrix} G_1(\lambda_0) & \cdots\cdots & G_1(\lambda_{n-1}) \\ \lambda_0 G_1(\lambda_0) & \cdots\cdots & \lambda_{n-1}G_1(\lambda_{n-1}) \\ \vdots & & \vdots \\ \lambda_0^{m_1-1} G_1(\lambda_0) & \cdots\cdots & \lambda_{n-1}^{m_1-1} G_1(\lambda_{n-1}) \\ \vdots & & \vdots \\ G_k(\lambda_0) & \cdots\cdots & G_k(\lambda_{n-1}) \\ \lambda_0 G_k(\lambda_0) & \cdots\cdots & \lambda_{n-1}G_k(\lambda_{n-1}) \\ \vdots & & \vdots \\ \lambda_0^{m_k-1} G_k(\lambda_0) & \cdots\cdots & \lambda_{n-1}^{m_k-1} G_k(\lambda_{n-1}) \end{pmatrix}$$

where $n = \Sigma_{i=1}^k m_i$.

<u>Proof:</u> The matrix M_n is nondegenerate for any choice of distinct λ_j, $j=0,\ldots,n-1$, if and only if for any vector a, $M_n a = 0$ implies $a = 0$. Let

$$a^T = (a_{1,0}, a_{1,1}, \ldots, a_{1,m_1-1}, \ldots, a_{k,0}, a_{k,1}, \ldots, a_{k,m_k-1}),$$

and let $P_\ell(x) = a_{\ell,0} + a_{\ell,1}x + \cdots + a_{\ell,m_\ell-1}x^{m_\ell-1}$, $\ell=1,\ldots,k$. Then $\sum_{\ell=1}^{k} P_\ell(x)\, G_\ell(x)$ vanishes at all the n distinct λ_j, since $M_n a = 0$. On the other hand, $\sum_{\ell=1}^{k} P_\ell(x)\, G_\ell(x)$ is a polynomial of degree less than n, hence must be identically zero. But then since all summands except for the ℓ-th one have the factor $(\alpha_\ell+\beta_\ell x)^{m_\ell}$, the ℓ-th summand must also have it and, since G_ℓ does not have it, P_ℓ must have it, and that is possible only when $P_\ell = 0$. This shows that $a = 0$, as required. $\square$

Remark 2.1: We choose distinct $\lambda_0,\ldots,\lambda_{n-1}$ with $\lambda_i \neq 1,0,1$, $\alpha_j + \beta_j\lambda_i \neq 0$, $j = 1,\ldots,k$, $i = 0,\ldots,n-1$, such that M_n in (2.1) is non-degenerate, and we denote this matrix with fixed λ_i as M_n^*. $\square$

We are ready to prove

Theorem 2.1: Let X be a k-direction mesh. Then

$$\dim S_\pi(X) = \Sigma_{1\leq i<j\leq k}\, m_i m_j. \tag{2.2}$$ $\square$

Proof: In the proof we denote the differential operator $Q_\ell(D)$ by Q_ℓ, $\ell=1,\ldots,k$. Since $S_\pi(X) = \mathfrak{D}(X)$, we need only to derive the

dimension of the space $\mathfrak{D}(X)$. Let Π_j be the linear space of all homogeneous polynomials of degree j. Observe that $\Pi_i \cap \Pi_j = \{0\}$ for $i \neq j$ and that $\{(x+\lambda_{0j}y)^j, (x+\lambda_{1j}y)^j, \ldots, (x+\lambda_{jj}y)^j\}$ is a basis of Π_j for arbitrary distinct λ_{ij}, $i=0,\ldots,j$, with $\lambda_{ij} \neq -1,0,1$.
Since $\mathfrak{D}(X)$ is a finite dimensional linear space of polynomials, $\mathfrak{D}(X)$ is a subspace of $\Pi_0 \oplus \cdots \oplus \Pi_N$ for sufficiently large N. Let $S_j = \mathfrak{D}(X) \cap \Pi_j$. Then $S_i \cap S_j = \{0\}$ for $i \neq j$, and therefore $S_0 \oplus \cdots \oplus S_N$ is well defined. We prove that

$$\mathfrak{D}(X) = S_0 \oplus \cdots \oplus S_N. \tag{2.3}$$

Indeed, $S_0 \oplus \cdots \oplus S_N \subseteq \mathfrak{D}(X)$ by the definition of S_j. To show that $\mathfrak{D}(X) \subseteq S_0 \oplus \cdots \oplus S_N$, take arbitrary $f \in \mathfrak{D}(X)$, and $f = \sum_{j=0}^{N} f_j$, where $f_j \in \Pi_j$. Due to (1.7), $Q_\ell f = \sum_{j=0}^{N} Q_\ell f_j = 0$, $\ell = 1,\ldots,k$. By (1.5), we know that for $i < j$ and $Q_\ell f_j \neq 0$, $\deg(Q_\ell f_i) < \deg(Q_\ell f_j)$. Thus $Q_\ell f_j = 0$, $\ell = 1,\ldots,k$, $j = 0,\ldots,N$, i.e., $f_j \in S_j$, $j = 0,\ldots,N$. This means that $\mathfrak{D}(X) \subseteq S_0 \oplus \cdots \oplus S_N$, which completes the proof of (2.3).

To derive dim $\mathfrak{D}$ we compute dim S_j, $j = 0,\ldots,N$, since $\dim \mathfrak{D}(X) = \sum_{j=0}^{N} \dim S_j$, due to (2.3). Let $f \in S_j$. Then

$$f(x,y) = \sum_{i=0}^{j} a_{ij}(x+\lambda_{ij}y)^j, \tag{2.4}$$

and

$$Q_\ell f = 0, \quad \ell = 1,\ldots,k. \tag{2.5}$$

Let $q_\ell = \deg Q_\ell = n - m_\ell$, where $n = \sum_{\ell=1}^{k} m_\ell$. Since

$$(2.6)\qquad \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)^m (x + \lambda y)^j$$

$$= \begin{cases} 0, & \text{if } m > j, \\ j(j-1)\cdots(j-m+1)(\alpha+\beta\lambda)^m (x+\lambda y)^{j-m}, & \text{if } m \le j, \end{cases}$$

from (2.4) we have

$$(2.7)\qquad Q_\ell f = \begin{cases} 0, & \text{if } q_\ell > j, \\ \sum_{i=0}^{j} a_{ij}\, j(j-1)\cdots(j-q_\ell+1) Q_\ell(1,\lambda_{ij})(x+\lambda_{ij}y)^{j-q_\ell}, & \\ & \text{if } q_\ell \le j. \end{cases}$$

Since $S_\pi(X)$ is the polynomial subspace of the linear space spanned by the translates of a box spline for which there are only n directions, polynomials in $S_\pi(X)$ have degree less than n. Thus we only need to derive $\dim S_j$, for $j=0,1,\ldots,n-1$.

From (2.7) we have

$$(2.8)\qquad \begin{cases} Q_\ell f = 0, & \text{if } q_\ell \ge j+1, \\ Q_\ell f = \sum_{i=0}^{j} a_{ij}\, j(j-1)\cdots(j-q_\ell+1) Q_\ell(1,\lambda_{ij})(x+\lambda_{ij}y)^{j-q_\ell}, & \\ & \text{if } q_\ell \le j. \end{cases}$$

From (2.5) and (2.8), we have a system of equation in a_{ij}:

$$(2.9) \qquad \sum_{i=0}^{j} a_{ij}\, Q_\ell(1,\lambda_{ij})\, \lambda_{ij}^{r} = 0,\ r=0,\ldots,j-q_\ell;\quad q_\ell \le j,$$

and the coefficient matrix M_j of (2.9) consists of blocks $B_{\ell,j}$:

$$B_{\ell,j} = \begin{pmatrix} Q_\ell(1,\lambda_{0j}) & \cdots\cdots\cdots & Q_\ell(1,\lambda_{jj}) \\ \lambda_{0j}Q_\ell(1,\lambda_{0j}) & \cdots\cdots\cdots & \lambda_{jj}Q_\ell(1,\lambda_{jj}) \\ \vdots & & \vdots \\ \lambda_{0j}^{j-q_\ell}\, Q_\ell(1,\lambda_{0j}) & \cdots\cdots & \lambda_{jj}^{j-q_\ell}\, Q_\ell(1,\lambda_{jj}) \end{pmatrix}$$

Since $j \le n-1$, $j-q_\ell = j-(n-m_\ell) = m_\ell-(n-j) \le m_\ell - 1$, M_j is a submatrix of M_n^* in Remark 2.1, and is contained in $\sum_{\ell=1}^{k}(j+1-q_\ell)_+$ rows. Notice that $j+1 \ge \sum_{\ell=1}^{k}(j+1-q_\ell)_+$, for $j \le n-1$, since we get equality when $j+1=n$. Since M_n^* is non-degenerate, we can find $j+1$ columns, such that the $\sum_{\ell=1}^{k}(j+1-q_\ell)_+$ by $(j+1)$ submatrix of M_n^*, corresponding to M_j, is of rank $\sum_{\ell=1}^{k}(j+1-q_\ell)_+$. Use the $j+1$ λ_i's in M_n^*, corresponding to the $j+1$ chosen columns as $\lambda_{0j},\ldots,\lambda_{jj}$

in (2.4) and (2.9), and M_j is obviously of rank

$\sum_{\ell=1}^{k} (j+1-q_\ell)_+$. Since the number of a_{ij}'s, $j+1$, is no less

than $\sum_{\ell=1}^{k} (j+1-q_\ell)_+$, the number of equations in (2.9), and

the coefficient matrix of (2.9), M_j ,is non-degenerate, the

solution space of (2.9) is of dimension

$(j+1) - \sum_{\ell=1}^{k} (j+1-q_\ell)_+$, which is the dimension of S_j. So

$$\sum_{j=0}^{n-1} \dim S_j = \sum_{j=0}^{n-1} [(j+1) - \sum_{\ell=1}^{k} (j+1-q_\ell)_+]$$

$$= \sum_{j=1}^{n} j - \sum_{\ell=1}^{k} \sum_{j=1}^{m_\ell} j = \sum_{1\le i<j\le k} m_i m_j,$$

since

$$\sum_{j=0}^{n-1} (j+1-q_\ell)_+ = 1 + \cdots + n-q_\ell \quad \text{and} \quad n-q_\ell = m_\ell. \qquad \Box$$

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References.

[1] C. de Boor, R. De Vore, Approximation by smooth multivariate splines, Trans. AMS 276 (1983) 775-788.

[2] C. de Boor, K. Hollig, Bivariate box splines and smooth pp. functions on a three direction mesh, J. Comp. and Applied Math., 9 (1983), pp. 13-28.

[3] C. de Boor, K. Hollig, B-splines from parallelepipeds, J. D'Anal. Math. 42 (1982/83) 99-115.

[4] W. Dahmen, C.A. Micchelli, On the optimal approximation rates for criss-cross finite element spaces, J. Comp. and Applied Math., 10 (1984) 255-273.

[5] W. Dahmen, C.A. Micchelli, Recent progress in multivariate splines, Approximation Theory IV, Edited by C.K. Chui, L.L. Schumaker, J.D. Ward, Academic Press, 1983.

[6] W. Dahmen, C.A. Micchelli, Some results on box splines, Bull. (New series) of the American Math. Society, Vol. 11, No. 1, July 1984.

[7] W. Dahmen, C.A. Micchelli, Translates of multivariate splines, Linear Algebra and its Applications, 52/53, 217-234 (1983).