Information Structures to Secure Control of Rigid Formations with Leader-Follower Structure

Tolga Eren, Walter Whiteley, Brian D.O. Anderson, A. Stephen Morse and Peter N. Belhumeur

***Abstract*— This paper is concerned with rigid formations of mobile autonomous agents using a leader-follower structure. A formation is a group of agents moving in real 2- or 3-dimensional space. A formation is called rigid if the distance between each pair of agents does not change over time under ideal conditions. Sensing/communication links between agents are used to maintain a rigid formation. Two agents connected by a sensing/communication link are called neighbors. There are two types of neighbor relations in rigid formations. In the first type, the neighbor relation is symmetric. In the second type, the neighbor relation is asymmetric. Rigid formations with a leader-follower structure have the asymmetric neighbor relation. A framework to analyze rigid formations with symmetric neighbor relations is given in our previous work. This paper suggests an approach to analyze rigid formations that have a leader-follower structure.**

I. Introduction

Multiagent systems have lately received considerable attention due to recent advances in computation and communication technologies (see for example [2], [8], [12], [13], [1], [7], [17], [9]). In the context of this paper, agents will simply be thought of as autonomous agents including robots, underwater vehicles, microsatellites, unmanned air vehicles, and ground vehicles. A *formation* is a group of agents moving in real 2- or 3-dimensional space. A formation is called *rigid* if the distance between each pair of agents does not change over time under ideal conditions. Sensing/communication links are used for maintaining fixed distances between agents. The interconnection structure of sensing/communication links is called *sensor/network topology*. In practice, actual agent groups cannot be expected to move exactly as a rigid formation because of sensing errors, vehicle modelling errors, etc. The ideal benchmark point formation against which the performance of an actual agent formation is to be measured is called a *reference formation*.

Two agents connected by a sensing/communication link are called *neighbors*. There are two types of neighbor relations in rigid formations. In the first type, the neighbor relation is symmetric, i.e., if agent i senses/communicates with agent j and uses the received information for motion planning, so does agent j with agent i. In the second type, the neighbor relation is asymmetric, i.e., if agent i senses/communicates with agent j and uses the received information for motion planning, then agent j does not make use of any information received from agent i although it may sense/communicate with agent i. For example, rigid formations with a leader-follower structure have the asymmetric neighbor relation. A link with an asymmetric neighbor relation between a leader and a follower is represented with a directed edge pointing from the follower to the leader. The terms, 'undirected formation' and 'directed formation', are also used to describe formations with symmetric neighbor relations and formations with leader-follower structure [12]. Eren et al. [4], [5], [3] and Olfati-Saber and Murray [8] suggested an approach based on rigidity for maintaining formations of autonomous agents with sensor/network topologies that use distance information between agents, where the neighbor relation is symmetric. For formations that have a leader-follower structure, Baillieul and Suri give two separate conditions for stable rigidity, one of which is a necessary condition and the other is a sufficient condition [1]. This paper suggests an approach to analyze rigid formations with a leader-follower structure and proves that the necessary condition given by Baillieul and Suri is a necessary and sufficient condition for stable rigidity in formations that have a leader-follower structure.

The paper is organized as follows. In §II, we start with definitions of point formations and rigidity. We then review rigid formations with symmetric neighbor relations in §III. We investigate rigid formations with leader-follower structure in §IV. We end the paper with concluding remarks in §V.

II. Rigidity and Point Formations

One way of visualizing rigidity is to imagine a collection of rigid bars connected to one another by idealized ball joints, which is called a bar-joint framework. By an idealized ball joint we mean a connection between a collection of bars which imposes only the restriction that the bars share common endpoints. Now, can the bars and joints be moved in a continuous manner without changing the

T. Eren and P. N. Belhumeur are with Department of Computer Science, Columbia University, 10027 New York, NY, USA, and they are supported by the National Science Foundation under grants NSF ITR IIS-00-85864, NSF EIA-02-24431 and NSF IIS-03-08185, eren|belhumeur@cs.columbia.edu

W. Whiteley is with the Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada, and he is supported by the Natural Science and Engineering Research Council (Canada) and the National Institutes of Health (USA), whiteley@mathstat.yorku.ca

B.D.O. Anderson is with the Australian National University and National ICT Australia. National ICT Australia is funded by the Australian Governments Department of Communications, Information Technology and the Arts and the Australian Research Council through Backing Australias Ability and the ICT Centre of Excellence Program, Brian.Anderson@nicta.com.au

A. S. Morse is with the Department of Electrical Engineering and Computer Science, Yale University, New Haven, CT, 06520, and he is supported by the National Science Foundation, as.morse@yale.edu

lengths of any of the bars, where translations and rotations do not count? If so, the framework is flexible; if not, it is rigid. (Precise definitions will appear in the sequel.) In a bar-joint framework, the length of a bar imposes a distance constraint for both end-joints. This is the same situation in a formation where two agents connected by a sensing/communication link are mutually affected by the information conveyed by this link. For example, if two agents connected by a sensing/communication link are set to maintain a ten meter distance between each other, then both agents perform action to maintain this distance. In the graph theoretic setting, the edge corresponding to this link is denoted by an undirected edge.

The situation in a rigid formation where the relation between agents has a leader-follower structure is different. Because the information on a sensing/communication link between a leader-follower pair is used only by the follower. For example, with the same distance requirement as in the example above, if two agents, labelled with i and j, are set to maintain a ten meter distance between themselves where i is the leader and j is the follower, then only agent j performs action to maintain this distance. Let us assume the following properties in a formation of agents: (i) there is a global formation leader that determines where the entire formation moves, and it does not follow any other member; (ii) there is a first-follower of the global leader that maintains a predefined distance only to the global leader; (iii) every other agent of the formation maintains predefined distances to some other agents in the formation; (iv) if an agent, say B, maintains a predefined distance to another agent, say A, then A does not perform any action to maintain a predefined distance to B (in this relation A is a leader and B is a follower). As the formation moves with the leadership of the global leader, if the distance between every pair of agents does not change over time under ideal conditions, then such a formation is a rigid formation.

Certain directed information patterns in a formation can be described by bar-joint frameworks. To do that, consider creating a bar-joint framework in the plane starting from two joints connected by a bar. Once the end-joints are held fixed (i.e., translations and rotations are avoided), we can insert a new joint by connecting it to the existing joints using new bars. In this scenario, the constraints imposed by the new bars act only on the newly inserted joint because the initial bar-joint framework is already fixed and cannot not be affected by the newly inserted bars and joints. If the resulting bar-joint framework is not deformable, then this new resulting bar-joint framework is rigid and it becomes the new fixed bar-joint framework for the next step. In the graph-theoretic setting, the directed edge points to the newly inserted joint from the fixed bar-joint framework.

To summarize, there are two types of neighbor relations. It can be symmetric, i.e., if agent i senses/communicates with agent j and performs action upon the information it receives, so does agent j. This corresponds to an undirected graph. Alternatively, the formation can have a leader-follower structure, i.e., agent j senses/communicates with agent i and performs actions upon the information it receives, but the actions of agent i do not depend on the information conveyed by the sensing/communication with agent j. The underlying graph of such formations is a directed graph. We will consider these two cases separately in each section.

A point formation $\mathbb{F}_p \triangleq (p, \mathcal{L})$ provides a way of representing a formation of n agents. $p \triangleq \{p_1, p_2, \ldots, p_n\}$ and the points p_i represent the positions of agents in $\mathbb{R}^d$ $\{d = 2$ or $3\}$ where i is an integer in $\{1, 2, \ldots, n\}$ and denotes the labels of agents. $\mathcal{L}$ is the set of "maintenance links," labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$. The *maintenance links* in $\mathcal{L}$ correspond to constraints between specific agents, such as distances, which are to be maintained over time by using sensing/communication links between certain pairs of agents. Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G}_{\mathbb{F}_p} \triangleq (\mathcal{V}, \mathcal{L})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$, which is the set of labels of agents, and edge set $\mathcal{L}$. A formation with distance constraints can be represented by $(\mathcal{V}, \mathcal{L}, f)$ where $f : \mathcal{L} \longmapsto \mathbb{R}$. Each maintenance link $(i, j) \in \mathcal{L}$ is used to maintain the distance $f((i, j))$ between certain pairs of agents fixed.

A *trajectory* of a formation is a continuously parameterized one-parameter family of curves $(q_1(t), q_2(t), ..., q_n(t))$ in $\mathbb{R}^{nd}$ which contain p and on which for each t, $\mathbb{F}_{q(t)}$ is a formation with the same measured values under f, g. A *rigid motion* is a trajectory along which point formations contained in this trajectory are congruent to each other. We will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_r$, where $p, r \in q(t)$, are congruent if they have the same graph and if p and r are congruent. p is *congruent* to r in the sense that there is a distance-preserving map $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $T(r_i) = p_i, i \in \{1, 2, \ldots, n\}$. If rigid motions are the only possible trajectories then the formation is called *rigid*; otherwise it is called *flexible* [4].

III. Rigidity in Point Formations with Symmetric Neighbor Relations

Whether a given point formation is rigid or not can be studied by examining what happens to the given point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ with m maintenance links, along the trajectory $q([0, \infty)) \triangleq \{\{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ on which the Euclidean distances $d_{ij} \triangleq ||p_i - p_j||$ between pairs of points (p_i, p_j) for which (i, j) is a link are constant. Along such a trajectory

$$(q_i - q_j) \cdot (q_i - q_j) = d_{ij}^2, \qquad (i, j) \in \mathcal{L}, \quad t \geq 0 \qquad (1)$$

We note that the existence of a trajectory is equivalent to the existence of a piecewise analytic path, with all derivatives at the initial point [11]. It is also equivalent to the existence of a sequence of formations on $p(n), n = 1, 2, \ldots$ with the same measurements, and with $\lim_{n\rightarrow\infty} p(n)$ converging to p. Assuming a smooth (piecewise analytic) trajectory, we can differentiate to get

$$(q_i - q_j) \cdot (\dot{q}_i - \dot{q}_j) = 0, \quad (i,j) \in \mathcal{L}, \quad t \geq 0 \quad (2)$$

Here, $\dot{q}_i$ is the velocity of point i. The m equations can be collected into a single matrix equation

$$R(q)\dot{q} = 0 \quad (3)$$

where $\dot{q} = \text{column } \{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\}$ and $R(q)$ is a specially structured $m \times dn$ matrix called the *rigidity matrix* [10], [15], [16].

Example 1. *Consider a planar point formation $\mathbb{F}_p$ shown in Figure 1. This has a rigidity matrix as shown in Table I.*

Let $\mathcal{M}_p$ be the manifold of points congruent to p. Because any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$, is one along which $\mathbb{F}_p$ undergoes rigid motion, (2) automatically holds along any trajectory which lies within $\mathcal{M}_p$. From this, it follows that the tangent space to $\mathcal{M}_p$ at p, written $\mathcal{T}_p$, must be contained in the kernel of $R(p)$. If the points $p_1, p_2, \ldots, p_n$ are in general position (which means that the points $p_1, p_2, \ldots, p_n$ do not lie on any hyperplane in $\mathbb{R}^n$), then $\mathcal{M}_p$ is $n(n+1)/2$ dimensional since it arises from the $n(n-1)/2$-dimensional manifold of orthogonal transformations of $\mathbb{R}^n$ and the n-dimensional manifold of translations of $\mathbb{R}^n$ [10]. Thus $\mathcal{M}_p$ is 6-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^3$, and 3-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^2$. We have $\text{rank } R(p) = nd-$ dimension kernel $R(q) \leq nd - n(n+1)/2$. The following theorem holds [10], [15]:

Theorem 2. *Assume $\mathbb{F}_p$ is a formation with at least d points in d-dimensional space $\{d = 2,$ or $3\}$ where* $\text{rank } R(p) = \max\{\text{rank } R(x) : x \in \mathbb{R}^d\}$. *$\mathbb{F}_p$ is rigid in $\mathbb{R}^d$ if and only if*

$$\text{rank } R(p) = \begin{cases} 2n-3 & \text{if } d = 2, \\ 3n-6 & \text{if } d = 3. \end{cases}$$

This theorem leads to the notion of the "generic" behavior of rigidity. When the rank is less than the maximum, the formation may still be rigid. However this type of rigidity lacks the generic behavior and thus is not addressed in this paper.

1) Generic Rigidity: We define a type of rigidity, called "generic rigidity," that is more useful for our purposes. It is possible to characterize generic rigidity in terms of the "generic rank" of R where by R's *generic* or maximal rank

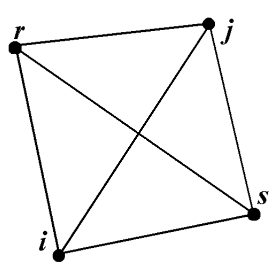

Fig. 1. A planar point formation.

we mean the largest value of $\text{rank}\{R(q)\}$ as q ranges over all values in $\mathbb{R}^{nd}$. The following theorem is due to Roth [10].

Theorem 3. *A formation $\mathbb{F}_p$ is generically rigid if and only if*

$$\text{generic rank } \{R(p)\} = \begin{cases} 2n-3 & \text{if } d = 2, \\ 3n-6 & \text{if } d = 3. \end{cases}$$

To understand this type of rigidity, it is useful to observe that the set of points p that satisfy the condition $\text{rank } R(p) = \max\{\text{rank } R(x) : x \in \mathbb{R}^d\}$ is a dense open subset of $\mathbb{R}^{nd}$ [10]. Thus, a generically rigid point formation $\mathbb{F}_p$ is rigid for almost all points in the neighborhood of points about p in $\mathbb{R}^{dn}$. The concept of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}_p$ but examines how well the rigidity of formations can be judged by knowing the vertices and their incidences, in other words, by knowing the underlying graph. For this reason, it is a desirable specialization of the concept of a "rigid formation" for our purposes. The following theorem holds for a generically rigid graph [15]:

Theorem 4. *The following are equivalent:*

1) *a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in d-dimensional space ($d = 2, 3$);*
2) *for some p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ is generically rigid;*
3) *for almost all p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ is generically rigid.*

A point formation $\mathbb{F}_p$ is *strongly generically rigid* if it is generically rigid and if $\text{rank } R(p) = \text{generic rank } \{R\}$. Hence, a strongly generically rigid formation is rigid and it remains rigid under small perturbations. This is the type of rigidity that is useful for our purposes.

As noted above, the concept of generic rigidity does not depend on the precise distances between the points in $\mathbb{F}_p$. For 2-dimensional space, we have a complete combinatorial characterization of generically rigid graphs, which was first proved by Laman in 1970 [6]. In the theorem below, $|\,.\,|$ is used to denote the cardinal number of a set.

Theorem 5 (Laman [6]). *A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in 2-dimensional space if and only if there is a subset $\mathcal{L}' \subseteq \mathcal{L}$ satisfying the following two conditions: (1) $|\mathcal{L}'| = 2|\mathcal{V}| - 3$, (2) For all $\mathcal{L}'' \subseteq \mathcal{L}', \mathcal{L}'' \neq \emptyset, |\mathcal{L}''| \leq 2|\mathcal{V}(\mathcal{L}'')| - 3$, where $|\mathcal{V}(\mathcal{L}'')|$ is the number of vertices that are end-vertices of the edges in $\mathcal{L}''$.*

2) Sequential Techniques: In this section, we present sequential techniques to create minimally rigid point formations. As noted earlier, Laman's Theorem characterizes rigidity in 2-dimensional space. But there is no comparable complete result for 3-dimensional space. Although we lack a characterization in 3-dimensional space, there are sequential techniques for generating rigid classes of graphs both

$R(p)$	i		j		r		s	
(i,j)	$x_i - x_j$	$y_i - y_j$	$x_j - x_i$	$y_j - y_i$	0	0	0	0
(i,r)	$x_i - x_r$	$y_i - y_r$	0	0	$x_r - x_i$	$y_r - y_i$	0	0
(i,s)	$x_i - x_s$	$y_i - y_s$	0	0	0	0	$x_s - x_i$	$y_s - y_i$
(j,r)	0	0	$x_j - x_r$	$y_j - y_r$	$x_r - x_j$	$y_r - y_j$	0	0
(j,s)	0	0	$x_j - x_s$	$y_j - y_s$	0	0	$x_s - x_j$	$y_s - y_j$
(r,s)	0	0	0	0	$x_r - x_s$	$y_r - y_s$	$x_s - x_r$	$y_s - y_r$

TABLE I
RIGIDITY MATRIX EXAMPLE FOR DISTANCES

in 2- and 3-dimensional space based on what are known as the vertex addition, edge splitting, and vertex splitting operations. First, we introduce the first two of these three operations, namely the vertex addition and edge splitting operations. Then we present sequences to create rigid point formations in which these operations are used. Before explaining these operations and sequences, we introduce some additional terminology.

If (i, j) is an edge, then we say that i and j are *adjacent* or that j is a *neighbor* of i and i is a neighbor of j. The vertices i and j are *incident* with the edge (i, j). Two edges are *adjacent* if they have exactly one common end-vertex. The *degree* or *valency* of a vertex i is the number of neighbors of i. If a vertex has k neighbors, it is called a *vertex of degree* k or a k*-valent vertex*.

One operation is the *vertex addition*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we add a new vertex i with d edges between i and d other vertices in $\mathcal{V}$ in d-dimensional space ($d = 2$, or 3). The other is the *edge splitting*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we remove an edge (j, k) in $\mathcal{L}$ and then we add a new vertex i with $d+1$ edges by inserting two edges (i, j), (i, k) and $d-1$ edges between i and $d-1$ vertices (other than j, k) in $\mathcal{V}$.

Now we are ready to present the following theorems:

Theorem 6 (vertex addition in undirected case - Tay, Whiteley [14]). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree d in d-dimensional space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ denote the subgraph obtained by deleting i and the edges incident with it. Then $\mathbb{G}$ is generically minimally rigid if and only if $\mathbb{G}^*$ is generically minimally rigid.*

Example 7. *Vertex addition operation in 2-dimensional space for an undirected graph is shown in Figure 2.*

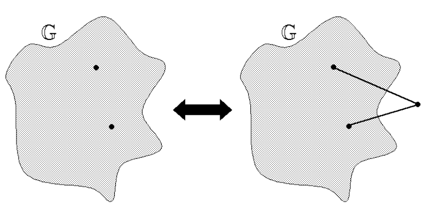

Fig. 2. Vertex addition - undirected case.

Theorem 8 (edge splitting in undirected case - Tay, Whiteley [14]). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of degree $d+1$; let $\mathcal{V}_i$ be the set of vertices incident to i; and let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ be the subgraph obtained by deleting i and its $d+1$ incident edges. Then $\mathbb{G}$ is generically minimally rigid if and only if there is a pair j, k of vertices of $\mathcal{V}_i$ such that the edge (j, k) is not in $\mathcal{L}^*$ and the graph $\mathbb{G}' = (\mathcal{V}^*, \mathcal{L}^* \bigcup (j, k))$ is generically minimally rigid.*

Example 9. *Edge splitting operation in 2-dimensional space for an undirected graph is shown in Figure 3.*

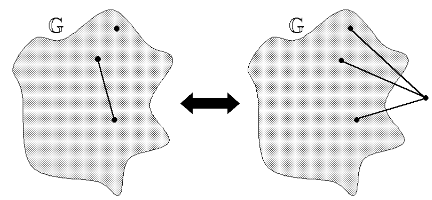

Fig. 3. Edge splitting - undirected case.

Vertex addition and edge splitting operations are used in Henneberg sequences.

Henneberg Sequences: Henneberg sequences are a systematic way of generating minimally rigid graphs based on the vertex addition and edge splitting operations [14]. In d-space, we are given a sequence of graphs: $\mathbb{G}_d, \mathbb{G}_{d+1}, \ldots, \mathbb{G}_{|\mathcal{V}|}$ such that:

1) $\mathbb{G}_d$ is the complete graph on d vertices;
2) $\mathbb{G}_{i+1}$ comes from $\mathbb{G}_i$ by adding a new vertex either by (i) the vertex addition or (ii) the edge splitting operation.

Note that $\mathbb{G}_i$ and $\mathbb{G}_{i+1}$ correspond to $\mathbb{G}^*$ and $\mathbb{G}$ in the statements of Theorem 6 and Theorem 8. All graphs in the sequence are minimally rigid in d-space.

IV. RIGIDITY IN POINT FORMATIONS WITH LEADER-FOLLOWER STRUCTURE

First, we give some definitions from graph theory, which are relevant to point formations with leader-follower structure. A graph in which each edge is replaced by a directed edge is called a *directed graph*, also called a *digraph*. When there is a danger of confusion, we will call a graph, which is not a directed graph, an *undirected graph*. A directed graph having no multiple edges or loops (corresponding to a binary adjacency matrix with 0's on the diagonal) is called a *simple directed graph*.

An *arc*, or *directed edge*, is an ordered pair of end-vertices. It can be thought of as an edge associated with a direction. Symmetric pairs of directed edges are called *bidirected edges*. We will use only directed graphs with no bidirected edges in the rest of the paper. The number of inward directed graph edges from a given graph vertex in a directed graph is called the *in-degree* of the vertex. The number of outward directed graph edges from a given graph vertex in a directed graph is called the *out-degree* of the vertex. A *cycle* of a graph $\mathbb{G}$ is a subset of the edge set of $\mathbb{G}$ that forms a path such that the first vertex of the path corresponds to the last. A *directed cycle* is an oriented cycle such that all arcs go the same direction. A digraph is *acyclic* if it does not contain any directed cycle.

In a formation with leader-follower structure, each link is denoted with a line directed from follower to leader. There is one global leader and one first-follower of the global leader. The global leader does not follow any other agent, and the first-follower only follows the global leader. They are connected with one link pointed from the first-follower to the global leader. The rest of the agents are followers of at least two other agents. They can also be leaders of other agents.

In a rigid formation with leader-follower structure, once we fix the positions of the global leader and the first-follower, the formation cannot deform, including translations and rotations. The global leader and the first-follower can make the entire rigid formation translate and rotate in 2-dimensional space by making maneuvers.

Recall that the first follower has a link of out-degree 1. Since each agent in rigid formation, except the global-leader and the first follower, has at least two links with an out-degree of 2, we expect at least $2(n-2)+1 = 2n-3$ links in total. We have the following conjecture:

Conjecture 10. *A formation with leader-follower structure is minimally rigid in 2-dimensional space if and only if the following two conditions hold for the underlying graph: (1) the undirected graph is minimally rigid; (2) the directed graph has exactly one vertex of out-degree 0, one vertex of out-degree 1, and the rest of the vertices are of out-degree 2.*

We call these two conditions *minimal rigidity conditions for directed graphs*.

3) Sequential Techniques: One operation is the *vertex addition*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we add a new vertex i of degree 2 with two edges between i and two other vertices in $\mathcal{V}$. The other is the *edge splitting*: given a minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$, we remove a directed edge (j, k) (directed from j to k) in $\mathcal{L}$ and then we add a new vertex i of degree 3 with three edges by inserting two edges (i, j), (i, k) and $d-1$ edges between i and one other vertex (other than j, k) in $\mathcal{V}$ such that the edge (i, j) is directed from j to i.

Now we are ready to present the following theorems:

Theorem 11 (vertex addition - directed case). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a digraph with a vertex i of out-degree 2 in 2-dimensional space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ denote the subgraph obtained by removing i and the edges incident with it. Then $\mathbb{G}$ satisfies the minimal rigidity conditions for directed graphs if and only if $\mathbb{G}^*$ satisfies the minimal rigidity conditions for directed graphs.*

Proof: Inserting/removing i from the undirected graph $\mathbb{G}$ is equivalent to vertex addition operation in undirected graphs. Undirected minimally rigid graphs maintain rigidity under vertex addition operation. Hence condition (1) is satisfied in both $\mathbb{G}$ and $\mathbb{G}^*$.

Now suppose that $\mathbb{G}$ satisfies the condition (2). If we remove i, then the out-degree of the vertices of $\mathbb{G}^*$ do not change. Similarly, suppose that $\mathbb{G}^*$ satisfies the condition (2). If insert i with out-degree 2, then the out-degree of the remaining vertices do not change. □

Example 12. *Vertex addition operation for a directed graph is shown in Figure 4.*

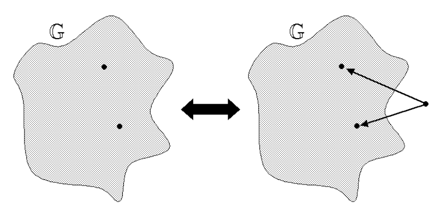

Fig. 4. Vertex Addition - directed case.

Theorem 13 (edge splitting - directed case). *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of out-degree 2 and in-degree 1 (where this edge is between i and j); let $\mathcal{V}_i$ be the set of vertices incident to i; and let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ be the subgraph obtained by deleting i and its three incident edges. Then $\mathbb{G}$ satisfies the minimal rigidity conditions for directed graphs if and only if there is a directed edge of a pair j, k (directed from j to k) of vertices of $\mathcal{V}_i$ such that the directed edge (j, k) is not in $\mathcal{L}^*$ and the graph $\mathbb{G}' = (\mathcal{V}^*, \mathcal{L}^* \bigcup (j, k))$ satisfies the minimal rigidity conditions for directed graphs.*

Proof: Condition (1) is the edge splitting operation for undirected graphs as explained in §III. Suppose that $\mathbb{G}$ satisfies condition (2). Then removing i only changes the out-degree of j. However, an edge is inserted from j to k. So all the vertices of $\mathbb{G}^*$ have out-degree of 2. Conversely, suppose that $\mathbb{G}^*$ satisfies condition (2). The newly inserted vertex i is of out-degree 2. These edges do not change the out-degree of other vertices. In the replacement of the edge (j, k) by (j, i), the out-degree of j is also preserved. □

Example 14. *Edge splitting operation for a directed graph is shown in Figure 5.*

For point formations with leader-follower structure, Baillieul and Suri define stably rigid formations as follows: a

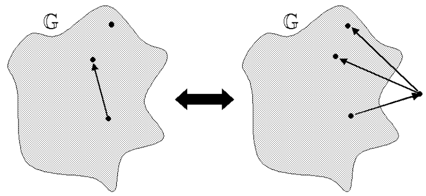

Fig. 5. Edge Splitting - directed case.

formation is *stably rigid* under a distributed relative distance control law as given in [1], if for any sufficiently small perturbation in the relative positions of the agents, the control law steers them asymptotically back into the prescribed formation in which the relative distance constraints are satisfied. They give the following theorem as a sufficiency condition for stably rigid formations:

Theorem 15 (Baillieul and Suri - Theorem 1 [1]). *If a formation is constructed from a single directed edge by a sequence of vertex addition operation, then it is stably rigid.*

Cycles in rigid formations that have a leader-follower structure are not desirable (see for example [1], [13]). Baillieul and Suri give the following proposition as a necessary condition for stably rigid formations:

Proposition 16 (Baillieul and Suri - Proposition 1 [1]). *If a formation with directed links is stably rigid then the following three conditions hold for the underlying graph: (i) the undirected underlying graph is generically minimally rigid; (ii) the directed graph is acyclic; (iii) the directed graph has no vertex with an out-degree greater than 2.*

Baillieul and Suri state that the conditions in Proposition 16 are not sufficient because there is a counterexample graph shown in Figure 6. In [1] it is stated that the graph satisfies the conditions of Proposition 16 but it is not stably rigid. However, we note that this graph actually does not satisfy the conditions of Proposition 16, because there is a cycle $(3, 5, 4, 6, 3)$ in the graph; hence it violates the condition ii of Proposition 16.

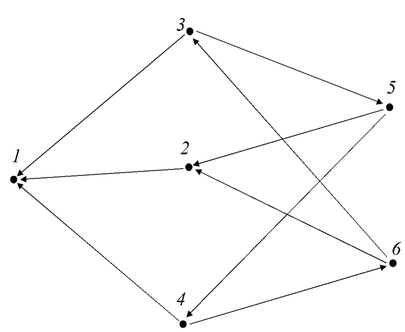

Fig. 6. The figure given by Baillieul and Suri in [1] as a counterexample.

It can be proved that the conditions given in Proposition 16 are also sufficient conditions; hence these conditions are necessary and sufficient conditions for stable rigidity. We have the following proposition:

Proposition 17. *A point formation in 2-dimensional space with directed links is stably rigid if and only if the following conditions hold for the underlying directed graph: (i) the undirected graph is generically minimally rigid; (ii) the directed graph is acyclic; (iii) the directed graph has no vertex with an out degree greater than 2.*

Proof: Baillieul and Suri prove the necessity part of the proof in [1]. Here we prove the sufficiency part only. Let us assume that the directed graph is acyclic. Then we can take the directed edges to define a partial order on the vertices: $a \geq b$ if the directed edge is pointed from a to b. We can extend this by transitivity. Since there are no cycles, this is a partial order with all vertices distinct. Since the graph is minimally generically rigid, all vertices have degree at least 2. Any maximal elements in this partial order have only outgoing edges - and therefore has two such edges. This can be removed (by reversed vertex addition operation) to give a smaller, minimally rigid graph satisfying all of the conditions. We continue this down to the initial directed edge (the global leader and the first follower). Since this reduction sequence can be reversed, the graph is constructed using only the vertex addition operation. By Theorem 15, such graphs are stably rigid. □

Corollary 18. *Equivalently a point formation in 2-dimensional space that has a leader-follower structure is stably rigid if and only if the point formation can be constructed from the initial edge by the vertex addition operation.*

The edge split operation is not used in [1] because it is stated in [1] that this operation results in vertices of degree 3. However, the edge split operation can be defined in such a way that the out-degrees of vertices remain less than 3. The definition given above for the edge split operation on directed minimally rigid graphs results vertices of degree 2. However, the edge splitting operation may or may not lead to cycles in the directed graph as shown in Figure 7. The question whether it can always be achieved so as to exclude any cycle or so as to include at least one cycle, depending on which we choose to aim for, is currently under investigation.

V. Concluding Remarks

In this paper, we suggested a way of analyzing rigid formations that have a leader-follower structure in 2-dimensional space. The necessity condition for stable rigidity is given in [1]. We proved that this condition is a necessary and sufficient condition for stable rigidity. Rigid formations that have a leader-follower structure in 3-dimensional space will be addressed in future work.

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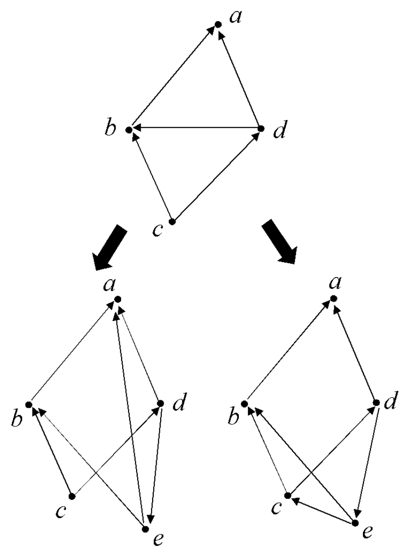

Fig. 7. Two examples of the edge splitting operation on a minimally rigid directed graph. The split edge is (d, b). We note that the resulting directed graph on the left has no cycles. On the other hand, the resulting directed graph on the right has a cycle (c,d,e,c). We note that the acyclic directed graph on the left can also be obtained by a series of vertex additions starting from a single edge.

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