Tractability of the Helmholtz equation with non-homogeneous Neumann boundary conditions: Relation to L_2-approximation

Arthur G. Werschulz*
Department of Computer and Information Sciences
Fordham University, New York, NY 10023
Department of Computer Science
Columbia University, New York, NY 10027

email: agw@cs.columbia.edu

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Abstract

We want to compute a worst case ε-approximation to the solution of the Helmholtz equation $-\Delta u + qu = f$ over the unit d-cube I^d, subject to Neumann boundary conditions $\partial_{\boldsymbol{\nu}} u = g$ on ∂I^d. Let card(ε, d) denote the minimal number of evaluations of f, g, and q needed to compute an absolute or normalized ε-approximation, assuming that f, g, and q vary over balls of weighted reproducing kernel Hilbert spaces. This problem is said to be weakly tractable if card(ε, d) grows subexponentially in ε^{-1} and d. It is said to be polynomially tractable if card(ε, d) is polynomial in ε^{-1} and d, and strongly polynomially tractable if this polynomial is independent of d. We have previously studied tractability for the homogeneous version $g = 0$ of this problem. In this paper, we investigate the tractability of the non-homogeneous problem, with general g. First, suppose that we use product weights, in which the role of any variable is moderated by its particular weight. We then find that if the sum of the weights is sublinearly bounded, then the problem is weakly tractable; moreover, this condition is more or less necessary. We then show that the problem is polynomially tractable if the sum of the weights is logarithmically or uniformly bounded, and we estimate the exponents of tractability for these two cases. Next, we turn to finite-order weights of fixed order ω, in which a d-variate function can be decomposed as a sum, each term depending on at most ω variables. We show that the problem is always polynomially tractable for finite-order weights, and we give estimates for the exponents of tractability. Since our results so far have established nothing stronger than polynomial tractability, we look more closely at whether strong polynomial tractability is possible. We show that our problem is never strongly polynomially tractable for the absolute error criterion. Moreover, we believe that the same is true for the normalized error criterion, but we have been able to prove this lack of strong tractability only when certain conditions hold on the weights. Finally, we use the Korobov and min kernels, along with product weights, to illustrate our results.

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1 Introduction

The *Helmholtz equation*

$$\mathbb{L}_q u := -\Delta u + qu = f \qquad \text{in } I^d := (0,1)^d, \tag{1}$$

is an important problem of applied mathematics, physics, and engineering. The function $q \in L_\infty(I^d)$ is bounded from below (almost everywhere) by some $q_0 \geq 0$. This problem can be solved subject to either *Dirichlet* boundary conditions

$$u = g \qquad \text{on } \partial I^d \tag{2}$$

or *Neumann* boundary conditions

$$\partial_{\boldsymbol{\nu}} u = g \qquad \text{on } \partial I^d, \tag{3}$$

where $\partial_{\boldsymbol{\nu}}$ is the outward-directed normal derivative.

How hard is it to solve this problem? Suppose that we measure error of an approximation in the $H^1(I^d)$-sense and that we use the worst case setting. We let $\mathrm{card}(\varepsilon, d, \Lambda)$ denote the minimal number of linear functionals of the problem data f, g, and q needed to obtain error of at most ε. The class Λ of linear functionals that we use in this paper will be either the class Λ^{all} of all linear functionals (*continuous linear information*) or the class Λ^{std} consisting of function values (*standard information*).

Let us momentarily consider a restricted version of this problem, in which $g = 0$ (*homogeneous* boundary conditions) and we have complete knowledge of q. For example, q may be fixed, having a particularly simple form (such as a constant). We also make the more-or-less standard assumption that f varies over the unit ball of the Sobolev space $H^r(I^d)$. Note that this version of the problem is *linear*, and so we can use the standard machinery that information-based complexity (IBC) provides for such problems, see [15, Section 4.5]. First, suppose that we use Λ^{all}. From [18, Chapter 5], we find that $\mathrm{card}(\varepsilon, d, \Lambda^{\mathrm{all}})$ is proportional to $(1/\varepsilon)^{d/(r+1)}$ for the restricted version of our problem. Next, suppose that we use Λ^{std}. Since the Sobolev embedding theorem tells us that standard information is not well-defined unless $r > d/2$, we will need to assume that f varies over the continuous functions belonging to the unit ball of $H^r(I^d)$ if $r \leq d/2$. Once again using [18, Chapter 5], along with [6], we find that if $r > d/2$, then $\mathrm{card}(\varepsilon, d, \Lambda^{\mathrm{std}})$ is proportional to $(1/\varepsilon)^{d/r}$, whereas if $r \leq d/2$, the ε-complexity is infinite for sufficiently small ε.

We now return to the original (non-restricted) problem, which is *nonlinear* because the solution u of (1) depends nonlinearly on q. Clearly, this problem is at least as hard as the restricted version. For the classical Sobolev formulation of our problem, the exponent of ε^{-1} can be arbitrarily large for fixed r and varying d. This means that our problem is *intractable*, since (using the terminology of Bellman [2]) it suffers from the *curse of dimensionality*.

If we want to vanquish the curse of dimensionality, we need to change the problem formulation. Since we are generally loath to give up the strong assurance of the worst case setting, we will need to assume that our problem data lie in spaces other than classical Sobolev spaces. *Weighted tensor product spaces* have been successfully used in the past as the source of input data for high-dimensional problems, see Chapter 5 ff. of [10], as well as the references contained therein.

In [20], we studied tractability for the Helmholtz equation under homogeneous Dirichlet or Neumann boundary conditions, showing that weighted tensor product spaces could snatch the Helmholtz problem from the jaws of intractability. It is only natural to ask whether these spaces can also help for non-homogeneous boundary value problems. In this paper, we will show that this is indeed the case for Neumann boundary conditions. Since the non-homogeneous Dirichlet problem needs different techniques than the non-homogeneous Neumann problem, we will treat the non-homogeneous Dirichlet problem in a future paper.

Here is a brief overview of this paper's contents.

The purpose of §2 is twofold. First, we precisely define the Neumann problem that we will be studying. We will assume that the problem data f, g and q belong to reproducing kernel Hilbert spaces (RKHSs). Let us make this idea more precise. For $\ell \in \{d-1, d\}$, let K_ℓ be a reproducing kernel defined over $[0,1]^{2\ell}$, and let $H(K_\ell)$ denote the resulting RKHS. We will let $H(\widetilde{K}_{d-1})$ be a RKHS of functions g defined on the boundary of I^d, such that the restriction of g to any face of I^d belongs to $H(K_{d-1})$. The function f will vary over the unit ball of $H(K_d)$. We will let q vary over the ρ-ball of $H(K_d)$, with the additional requirement that $q \geq q_0$. Finally, g will vary over those elements of the unit ball of $H(\widetilde{K}_{d-1})$.

The second purpose of §2 is to precisely define what we mean by an ε-approximation, which will depend on the *error criterion* used. On the one hand, we can use the *absolute* error criterion, in which we want the error in our

solution to be at most ε; on the other hand, we can use the *normalized* error criterion, in which we want to reduce the initial error by a factor of ε. Here, the *initial error* is the minimal error over all algorithms using no information whatsoever about the problem data f, g, and q, which turns out to be the error of the zero algorithm. Although the absolute error criterion can be influenced (for good or ill) by the scaling of our problem, the normalized error criterion is less sensitive to such concerns. Obviously, $\mathrm{card}(\varepsilon, d, \Lambda)$ depends on the error criterion being used. We stress that $\mathrm{card}(\varepsilon, d\Lambda)$ only determines the *information complexity*, i.e., the amount of information needed. Clearly, it is important to study the *total complexity*, which also includes the combinatory cost of using information about the problem data to obtain an approximation. Since our problem is nonlinear, it is not clear whether the total complexity is of the same order as the information complexity. We hope to investigate this issue in a future paper.

§3 gives some a priori inequalities for our problem. Perhaps the most important is a perturbation estimate for the difference between two solutions corresponding to different problem data. This estimate relies on the Sobolev trace theorem. Since the usual proofs of this theorem involve partitions of unity, they suffer from several disadvantages:

- They require more smoothness than that provided by the unit cube.
- They give no clue about how the embedding constant depends on the domain in question.
- They tend to be somewhat complicated.

However, Vilmos Komornik [7] developed a simple, elegant proof of the trace theorem in 2003, which allows one to explicitly compute the embedding constant for domains (such as the unit cube) whose boundary is only piecewise C^1. Since his result is only available via the World Wide Web, Prof. Komornik has graciously allowed me to include his proof in this paper.

In §4, we show that if we know how to do L_2-approximation for functions defined over a unit cube, then we can approximate the solution of our problem. We show that for a fixed value of d, the information complexity of our Neumann problem is dominated by the information complexity of the L_2-approximation problem.

In §5, we discuss various notions of tractability, see [10] and the references cited therein. Our problem is said to be

- *weakly tractable* if $\mathrm{card}(\varepsilon, d, \Lambda)$ grows subexponentially in ε^{-1} and d,
- *polynomially tractable* if $\mathrm{card}(\varepsilon, d, \Lambda)$ grows polynomially in ε^{-1} and d, and
- *strongly polynomially tractable* if $\mathrm{card}(\varepsilon, d, \Lambda)$ grows polynomially in ε^{-1}, independent of d.

Since the information complexity tacitly depends on the error criterion being used, the same is true about the tractability of the problem. Although generalized (i.e., not necessarily polynomial) tractability has recently been studied in [5], the vast majority of work on tractability has dealt with polynomial tractability. Since this paper deals only with polynomial tractability and strong polynomial tractability, we will omit the adjective "polynomial" in the sequel whenever this will cause no confusion.

So far, the reproducing kernels determining our problem elements have been more or less arbitrary. If we are going to discuss tractability, these kernels must be related to each other in some manner as d varies. In §6, we discuss *weighted* RKHSs. The idea here is that we start out with a fixed "master" kernel K for the univariate case, such as the Korobov kernel (29) or the min kernel (30). We introduce a family $\boldsymbol{\gamma} = \{\, \gamma_{d,\mathfrak{u}} : \mathfrak{u} \subseteq \{1, 2, \ldots, d\}, d \in \mathbb{Z}^+ \,\}$ of weights. The most well-studied weights have been

- *product weights* (31), in which the role of any given variable is moderated by its particular weight, and
- *finite-order weights* (32) of *order* ω, in which our d-variate functions can be decomposed as sums, each term being a function of at most ω variables.

Our reproducing kernel K_d is then

$$K_d(\mathbf{x}, \mathbf{y}) = \sum_{\mathfrak{u} \subseteq \{1,2,\ldots,d\}} \gamma_{d,\mathfrak{u}} \prod_{j \in \mathfrak{u}} K(x_j, y_j) \qquad \forall\, \mathbf{x}, \mathbf{y} \in \bar{I}^d.$$

Hence tractability now depends on the weight sequence $\boldsymbol{\gamma}$ and master kernel K.

In §7, we give tractability results for weighted RKHSs, these results holding for any master kernel K.

1. First, we look at product weights. We find that if the sum of the weights is sublinearly bounded in d, then the Neumann problem is weakly tractable for both Λ^{all} and Λ^{std}, under both the absolute and normalized error criteria. Moreover, if the sum of the weights is *not* sublinearly bounded in d, then there is a kernel K such that the problem is not weakly tractable under the absolute error criterion. We then show that if the sum of the weights is logarithmically bounded in d, then the problem is tractable for both the absolute and normalized error criteria, giving estimates for the exponents of tractability.

2. We next look at finite-order weights, assuming that the weights themselves are uniformly bounded. Under these conditions, we prove that the problem is always tractable for both the absolute and normalized error criteria, giving estimates for the exponents of tractability.

The results of §7 only establish tractability, and not strong tractability. This should be contrasted with the results of [20], where we were able to show that the homogeneous Dirichlet and Neumann problems are strongly tractable under certain conditions on the weights. Is our lack of a strong tractability result for the nonhomogeneous Neumann problem an artifact of our proof techniques, or is it inherently characteristic of our problem? Since I^d has $2d$ faces, one would expect that any "good" algorithm for this problem will need to sample at each face of I^d; this would rule out the possibility of strong tractability. We make this precise in §8. Regardless of whether we are using continuous linear information or standard information, the nonhomogeneous Neumann problem *cannot* be strongly tractable for the absolute error criterion. We conjecture that this is also true for the normalized error criterion, but we have only been able to prove this under certain hypotheses on the weights.

The results so far have been for a more-or-less arbitrary master reproducing kernel K. In §9, we look at two specific kernels: the Korobov kernel (29) and the min kernel (30). We restrict our attention to product weights. The recent paper [9] considers the resulting L_2-approximation problem, giving estimates on the exponents of tractability depending on the kernel and the weights. Using these results, we derive estimates on the exponents of tractability for the Neumann problem.

This is not the whole story for the Neumann problem; we have additional results. Since the current paper is already so lengthy, they will be included in a separate paper. However, let us give the reader a brief overview of these results.

The results in the current paper are based on L_2-approximation. We can get better results if we use L_2-approximation for f and g, and L_∞-approximation for q. The results of [8, 9] give conditions under which L_2- and L_∞-approximation are related, which are illustrated for the Korobov and min kernels under product weights. We can use these results to get better estimates for the tractability exponents of the non-homogeneous Neumann problem. In addition, we will revisit strong tractability for this problem. We will show that the non-homogeneous Neumann problem is strongly tractable for a slightly-reformulated definition of Λ^{all} under the normalized error criterion.

2 Problem definition

In this section, we define the Neumann problem to be studied. Having done so, we then recall some basic concepts of IBC.

Let us establish a few notational conventions. If R is an ordered ring, then R^+ and R^{++} respectively denote the non-negative and positive elements of R. If X and Y are normed linear spaces, then $\text{Lin}[X, Y]$ denotes the space of bounded linear transformations of X into Y. We write $\text{Lin}[X]$ for $\text{Lin}[X, X]$, and X^* for $\text{Lin}[X, \mathbb{R}]$. The unit interval $(0, 1)$ is denoted by I. Finally, we use the standard notation for Sobolev inner products, seminorms, norms, and spaces, found in, e.g., [11, 18].

We first start with a variational formulation of the Helmholtz equation (1) under the Neumann boundary conditions (3), see (e.g.) [3, pp. 35–40]. Let $\mathbb{L}_q = -\Delta + q$, as in (1) and

$$B_d(v, w; q) = \int_{I^d} [\nabla v \cdot \nabla w + q v w] \qquad \forall\, v, w \in H^1(I^d),\ q \in L_\infty(I^d),$$

so that

$$B_d(v, w; q) = \langle \mathbb{L}_q v, w\rangle_{L_2(I^d)} + \langle \partial_{\boldsymbol{\nu}} v, w\rangle_{L_2(\partial I^d)} \qquad \forall\, v \in H^2(I^d), w \in H^1(I^d). \tag{4}$$

Let q_0 be a positive number, independent of d. Define

$$Q_d = \{\, q \in L_\infty(I^d) : q \geq q_0 \,\}.$$

The Lax-Milgram lemma implies that for $f \in L_2(I^d)$, $g \in L_2(\partial I^d)$, and $q \in Q_d$, there exists a unique $u = S_d(f, g, q) \in H^1(I^d)$ such that

$$B_d(u, w; q) = \langle f, w \rangle_{L_2(I^d)} + \langle g, w \rangle_{L_2(\partial I^d)} \qquad \forall\, w \in H^1(I^d). \tag{5}$$

From (4), we see that u is the variational solution to the Neumann problem (1) and (3).

We will want to approximate $S_d(f, g, q)$ for f, g, and q belonging to certain reproducing kernel Hilbert spaces (RKHSs), which we shall now define.

For $d \in \mathbb{Z}^{++}$, let K_d be a reproducing kernel defined over $\bar{I}^{\,2d}$, with $H(K_d)$ denoting the resulting RKHS, see (e.g.) [1] for further discussion. The norm and inner product of $H(K_d)$ will be respectively denoted by $\langle \cdot, \cdot \rangle_{H(K_d)}$ and $\| \cdot \|_{H(K_d)}$. In what follows, we assume that

$$\operatorname*{ess\,sup}_{x \in I^d} |K_d(\mathbf{x}, \mathbf{x})| < \infty \qquad \forall\, d \in \mathbb{Z}^{++}. \tag{6}$$

It then follows that $H(K_d)$ is continuously embedded in both $L_2(I^d)$ and $L_\infty(I^d)$. More precisely, let $\mathrm{App}_{d,p}$ denote the embedding of $H(K_d)$ into $L_p(I^d)$ for $p \in \{2, \infty\}$ defined by

$$\mathrm{App}_{d,p}\, v = v \qquad \forall\, v \in H(K_d).$$

(We use $\mathrm{App}_{d,p}$ as the name of this embedding, since we will be discussing the L_p-approximation of functions from $H(K_d)$ in the sequel.) From the reproducing property of K_d, we find that

$$\| \mathrm{App}_{d,2} \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]} \leq \left(\int_{I^d} K_d(\mathbf{x}, \mathbf{x})\, d\mathbf{x} \right)^{1/2} \tag{7}$$

and

$$\| \mathrm{App}_{d,\infty} \|_{\mathrm{Lin}[H(K_d), L_\infty(I^d)]} \leq \operatorname*{ess\,sup}_{\mathbf{x} \in I^d} |K_d(\mathbf{x}, \mathbf{x})|. \tag{8}$$

Hence, the embeddings $\mathrm{App}_{d,2}$ and $\mathrm{App}_{d,\infty}$ are well-defined continuous linear mappings, as claimed.

We also need a space of boundary-value functions. This will be the space $H(\widetilde{K}_{d-1})$, which consists of functions defined over ∂I^d whose restrictions to any face of I^d belongs to $H(K_{d-1})$.

First, we consider the case $d \geq 2$. For $j \in \{1, \ldots, d\}$ and $\theta \in \{0, 1\}$, let $I^d_{j,\theta}$ denote the face $x_j = \theta$ of the unit d-cube, so that

$$\partial I^d = \bigcup_{\substack{1 \leq j \leq d \\ \theta \in \{0,1\}}} I^d_{j,\theta}.$$

Then we define

$$H(\widetilde{K}_{d-1}) = \left\{ \partial I^d \xrightarrow{v} \mathbb{R} : v\big|_{I^d_{j,\theta}} \in H(K_{d-1}) \text{ for } j \in \{1, \ldots, d\}, \theta \in \{0, 1\} \right\},$$

which is a Hilbert space under the inner product

$$\langle v, w \rangle_{H(\widetilde{K}_{d-1})} := \sum_{\substack{1 \leq j \leq d \\ \theta \in \{0,1\}}} \left\langle v\big|_{I^d_{j,\theta}}, w\big|_{I^d_{j,\theta}} \right\rangle_{H(K_{d-1})} \qquad \forall\, v, w \in H(\widetilde{K}_{d-1}).$$

By our choice of notation, we are hinting that $H(\widetilde{K}_{d-1})$ is an RKHS under a reproducing kernel $\widetilde{K}_{d-1}$. We now define this reproducing kernel. Let $\mathbf{x}, \mathbf{y} \in \partial I^d$, so that $\mathbf{x} \in I^d_{j,\theta}$ and $\mathbf{y} \in I^d_{j',\theta'}$ for some $j, j' \in \{1, \ldots, d\}$ and $\theta, \theta' \in \{0, 1\}$. Then

$$\widetilde{K}_{d-1}(\mathbf{x}, \mathbf{y}) = \delta_{j,j'} \delta_{\theta,\theta'} K_{d-1}(\hat{\mathbf{x}}_{[j]}, \hat{\mathbf{y}}_{[j']}),$$

where

$$\hat{\mathbf{x}}_{[j]} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \qquad \text{and} \qquad \hat{\mathbf{y}}_{[j']} = (y_1, \ldots, y_{j'-1}, y_{j'+1}, \ldots, y_d).$$

In other words, if $\mathbf{x} \in I^d_{j,\theta}$, then

$$\widetilde{K}_{d-1}(\mathbf{x}, \cdot) = \begin{cases} K_{d-1}(\hat{\mathbf{x}}_{[j]}, \cdot) & \text{on } I^d_{j,\theta}, \\ 0 & \text{otherwise.} \end{cases}$$

We see that $H(\widetilde{K}_{d-1})$ is continuously embedded in both $L_2(\partial I^d)$ and $L_\infty(\partial I^d)$. More precisely, for $p \in \{2, \infty\}$, let $\widetilde{\mathrm{App}}_{d-1,p}$ denote the embedding of $H(\widetilde{K}_{d-1})$ into $L_p(\partial I^d)$. Since

$$\left(\widetilde{\mathrm{App}}_{d-1,p} g\right)\big|_{I^d_{j,\theta}} = \mathrm{App}_{d-1,p}\left(g\big|_{I^d_{j,\theta}}\right)$$

for any $g \in H(\widetilde{K}_{d-1})$ and any face $I^d_{j,\theta}$ of I^d, we may use (7) and (8) to see that

$$\|\widetilde{\mathrm{App}}_{d-1,2}\|_{\mathrm{Lin}[H(\widetilde{K}_{d-1}), L_2(\partial I^d)]} = \|\,\mathrm{App}_{d-1,2}\,\|_{\mathrm{Lin}[H(K_{d-1}), L_2(I^{d-1})]} \le \left(\int_{I^{d-1}} K_{d-1}(\mathbf{x}, \mathbf{x})\, d\mathbf{x}\right)^{1/2} \tag{9}$$

and

$$\|\widetilde{\mathrm{App}}_{d-1,\infty}\|_{\mathrm{Lin}[H(\widetilde{K}_{d-1}), L_\infty(\partial I^d)]} = \|\,\mathrm{App}_{d-1,\infty}\,\|_{\mathrm{Lin}[H(K_{d-1}), L_\infty(I^{d-1})]} \le \operatorname*{ess\,sup}_{\mathbf{x} \in I^d} |K_{d-1}(\mathbf{x}, \mathbf{x})|.$$

Up to this point, we have defined the kernel $\widetilde{K}_{d-1}$ for the case $d \ge 2$. What should we do when $d = 1$, i.e., how should we define the kernel $\widetilde{K}_0$? Since the Neumann boundary conditions for can be recovered exactly, we can take $\widetilde{K}_0$ to be the identity operator.

We want to efficiently compute approximations of $S_d(f, g, q)$ for $[f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho})$. Here, H_d and $\widetilde{H}_{d-1}$ are the respective unit balls in $H(K_d)$ and $H(\widetilde{K}_{d-1})$, and $H_{d,\rho}$ is the ball of radius ρ in $H(K_d)$, where $\rho \in \mathbb{R}^{++}$ is independent of d.

Of course, this problem is well-defined if and only if $Q_d \cap H_{d,\rho}$ is nonempty. In particular, we will assume that the positive constant function q_0 belongs to $H_{d,\rho}$.

Let $U_{d,n}$ be an algorithm using at most n information evaluations from a class Λ of linear functionals on $H(K_d) \cup H(K_{d-1})$. Here, Λ is either $\Lambda^{\mathrm{all}} = [H(K_d)]^* \cup [H(K_{d-1})]^*$ (*continuous linear information*), or the class Λ^{std} consisting of function evaluations on $\bar{I}^d$ (*standard information*).

Remark 2.1. How should we count these information evaluations? For standard information, which has the form

$$N([f, g, q]) = [f(\mathbf{x}_1), \ldots, f(\mathbf{x}_{n_1}), g(\mathbf{y}_1), \ldots, g(\mathbf{y}_{n_2}), q(\mathbf{z}_1), \ldots, q(\mathbf{z}_{n_3})] \qquad \forall [f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho}),$$

the answer is straightforward: the total number of information evaluations is $n_1 + n_2 + n_3$. Things get a bit trickier when we deal with continuous linear information, which has the form

$$N([f, g, q]) = [N_1 f, N_2 g, N_3 q] \qquad \forall [f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho}), \tag{10}$$

where

$$\begin{aligned} N_1 f &= [\langle f, f_1\rangle_{H(K_d)}, \ldots, \langle f, f_{n_1}\rangle_{H(K_d)}] && \forall f \in H_d, \\ N_2 g &= [\langle g, g_1\rangle_{H(\widetilde{K}_{d-1})}, \ldots, \langle g, g_{n_2}\rangle_{H(\widetilde{K}_{d-1})}] && \forall g \in \widetilde{H}_{d-1}, \\ N_3 q &= [\langle q, q_1\rangle_{H(K_{d-1})}, \ldots, \langle q, q_{n_3}\rangle_{H(K_{d-1})}] && \forall q \in Q_d \cap H_{d,\rho}. \end{aligned} \tag{11}$$

Clearly the total number of information evaluations needed to calculate $N_1 f$ and $N_3 q$ are n_1 and n_3, respectively. However, we need to be more careful when counting the number of information evaluations needed to calculate $N_2 g$. This is because we are counting $H(K_{d-1})$-inner products as primitive operations, rather than $H(\widetilde{K}_{d-1})$-inner products. Writing the ith information evaluation appearing in $N_2 g$ as

$$\langle g, g_i\rangle_{H(\widetilde{K}_{d-1})} = \sum_{\substack{1 \le j \le d \\ \theta \in \{0,1\}}} \left\langle g\big|_{I^d_{j,\theta}}, g_i\big|_{I^d_{j,\theta}} \right\rangle_{H(K_{d-1})},$$

we see that we can evaluate $\langle g, g_i \rangle_{H(\widetilde{K}_{d-1})}$ using k_i evaluations of $H(K_{d-1})$-inner products, where $k_i \in \{1, \ldots, 2d\}$ is the number of I^d-faces at which g_i is not identically zero. This means that we can evaluate the information (10)–(11) using

$$n_1 + \sum_{i=1}^{n_2} k_i + n_3 \le n_1 + 2d\, n_2 + n_3$$

information evaluations. □

The worst case *error* of $U_{d,n}$ is given by

$$e(U_{d,n}, S_d, \Lambda) = \sup_{[f,g,q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho})} \|S_d(f, g, q) - U_{d,n}(f, g, q)\|_{H^1(I^d)}$$

and the nth *minimal error* is defined to be

$$e(n, S_d, \Lambda) = \inf_{U_{d,n}} e(U_{d,n}, S_d, \Lambda),$$

the infimum being over all algorithms using at most n information evaluations from Λ. Since the zeroth minimal error uses no information evaluations at all, it is independent of Λ, and so we simply write it as $e(0, S_d)$, rather than $e(0, S_d, \Lambda)$.

If $\varepsilon \in (0, 1)$, we say that the algorithm $U_{d,n}$ provides an ε-*approximation* to S_d if

$$e(U_{d,n}, S_d, \Lambda) \le \varepsilon \cdot \mathrm{ErrCrit}(S_d).$$

Here, ErrCrit will be one of the two error criteria

$$\mathrm{ErrCrit}(S_d) = \begin{cases} 1 & \text{for absolute error,} \\ e(0, S_d) & \text{for normalized error.} \end{cases} \tag{12}$$

Let

$$\mathrm{card}(\varepsilon, S_d, \Lambda) = \min\{\, n \in \mathbb{Z}^+ : e(n, S_d, \Lambda) \le \varepsilon \cdot \mathrm{ErrCrit}(S_d) \,\}$$

denote the ε-*cardinality number*, i.e., the minimal number of information evaluations from Λ needed to compute an ε-approximation to S_d. In what follows, we shall let denote the ε-cardinality numbers for the absolute and normalized error criteria by "$\mathrm{card}^{\mathrm{abs}}$" and "$\mathrm{card}^{\mathrm{nor}}$", respectively. We shall write "card" when we are dealing with results that apply to either error criterion. Note that $\mathrm{card}(\varepsilon, S_d, \Lambda)$ was denoted $\mathrm{card}(\varepsilon, d, \Lambda)$ in the Introduction. The reason for writing S_d instead of d is that it will be helpful to stress the specific problem that we are considering in the sequel.

3 Some a priori estimates

In this section, we establish some a priori estimates that will be useful later. Our main goal to establish Lipschitz continuity of our problem. More precisely, we will show that if $[f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho})$ and $[\tilde{f}, \tilde{g}, \tilde{q}] \in H_d \times \widetilde{H}_{d-1} \times H_{d,\rho}$, then

$$\left\| S_d(f, g, q) - S_d\big(\tilde{f}, \tilde{g}, \phi(\tilde{q})\big) \right\|_{H^1(I^d)} \le C_{d,\mathrm{Lip}} \left(\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \tilde{q}\|_{L_2(I^d)} + \|g - \tilde{g}\|_{L_2(\partial I^d)} \right),$$

with an explicit value for the Lipschitz constant $C_{d,\mathrm{Lip}}$ that depends on the parameters describing our problem. Here, $\phi\colon H(K_d) \to Q_d$ is defined as

$$\phi(v)(\mathbf{x}) = \max\{v(\mathbf{x}), q_0\} \qquad \forall\, \mathbf{x} \in I^d,\ v \in H(K_d).$$

We proceed in several steps.

Our first step is to give a trace theorem for the unit cube that gives an explicit value for the embedding constant. As mentioned in the Introduction, this proof was discovered by Vilmos Komornik [7], and is presented here with his permission.

First, we give the trace inequality for general regions Ω.

Theorem 3.1. *Let Ω be a bounded open domain of $\mathbb{R}^d$ having a piecewise-C^1 boundary and outer-directed unit normal $\boldsymbol{\nu}$. Suppose that there exists a function $\mathbf{h}\colon \overline{\Omega} \to \mathbb{R}^d$ that is continuously differentiable on Ω such that*

$$\alpha := \inf_{\mathbf{x}\in\partial\Omega} \mathbf{h}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) > 0.$$

Let

$$C_{\text{emb}}(\Omega) = \sqrt{\frac{\|\mathbf{h}\|_{L_\infty(\Omega)} + \|\operatorname{div}\mathbf{h}\|_{L_\infty(\Omega)}}{\alpha}}.$$

Then $H^1(\Omega)$ is continuously embedded in $L_2(\partial\Omega)$, with

$$\|v\|_{L_2(\partial\Omega)} \le C_{\text{emb}}(\Omega)\|v\|_{H^1(\Omega)} \qquad \forall\, v \in H^1(\Omega).$$

Proof. By the usual density argument, it suffices to establish the estimate for all $v \in C^1(\overline{\Omega})$. For such v, we have

$$\operatorname{div}(v^2\mathbf{h}) = \nabla(v^2)\cdot\mathbf{h} + v^2(\operatorname{div}\mathbf{h}) = 2v\nabla v\cdot\mathbf{h} + v^2(\operatorname{div}\mathbf{h}).$$

Using the divergence theorem and the definition of α, we now find that

$$\alpha\int_{\partial\Omega} v^2 \le \int_{\partial\Omega}(\mathbf{h}\cdot\boldsymbol{\nu})v^2 = \int_{\partial\Omega}(v^2\mathbf{h})\cdot\boldsymbol{\nu} = \int_\Omega \operatorname{div}(v^2\mathbf{h}) = 2\int_\Omega v\,(\nabla v\cdot\mathbf{h}) + \int_\Omega v^2(\operatorname{div}\mathbf{h}).$$

Using this inequality, along with the bounds

$$2\int_\Omega v\,(\nabla v\cdot\mathbf{h}) \le \|\mathbf{h}\|_{L_\infty(\Omega)}\left(2\,\|v\|_{L_2(\Omega)}\|\nabla v\|_{L_2(\Omega)}\right) \le \|\mathbf{h}\|_{L_\infty(\Omega)}\left(\|v\|^2_{L_2(\Omega)} + \|\nabla v\|^2_{L_2(\Omega)}\right)$$

and

$$\int_\Omega v^2(\operatorname{div}\mathbf{h}) \le \|\operatorname{div}\mathbf{h}\|_{L_\infty(\Omega)}\|v\|^2_{L_2(\Omega)},$$

we obtain the desired result. □

Specializing to the case $\Omega = I^d$, we have

Corollary 3.1. *The space $H^1(I^d)$ is continuously embedded in $L_2(\partial I^d)$, with*

$$\|v\|_{L_2(\partial I^d)} \le \sqrt{2d+1}\,\|v\|_{H^1(I^d)} \qquad \forall\, v \in H^1(I^d).$$

Proof. Choose $\mathbf{h}(\mathbf{x}) := \mathbf{x} - \frac{\mathbf{1}}{\mathbf{2}}$ in Theorem 3.1. If we let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard basis vectors of $\mathbb{R}^d$, we find that

$$\boldsymbol{\nu}\big|_{I^d_{j,\theta}} = (-1)^{\theta+1}\mathbf{e}_j \qquad \text{for } j \in \{1,\ldots,d\},\ \theta \in \{0,1\}.$$

Thus

$$\alpha = \inf_{\mathbf{x}\in\partial\Omega}\mathbf{h}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) = \tfrac{1}{2} \qquad \text{and} \qquad \|\mathbf{h}\|_{L_\infty(\Omega)} + \|\operatorname{div}\mathbf{h}\|_{L_\infty(\Omega)} = \tfrac{1}{2} + d,$$

and so $C_{\text{emb}}(I^d) \le \sqrt{2d+1}$. □

Our next step is to use a minimum principle to establish an upper bound on the L_∞-norm of the solution.

Lemma 3.1. *Let*

$$\eta_0 = \frac{1}{q_0} \qquad \textit{and} \qquad \eta_1 = \frac{1+\cosh\sqrt{q_0}}{\sqrt{q_0}\cosh\sqrt{q_0}}. \tag{13}$$

For $f \in L_2(I^d)$, $g \in L_2(\partial I^d)$, and $q \in Q_d$, we have

$$\|S_d(f,g,q)\|_{L_\infty(I^d)} \le \eta_0\|f\|_{L_\infty(I^d)} + \eta_1 d\|g\|_{L_\infty(\partial I^d)}.$$

Proof. In the proof of this lemma, all pointwise inequalities are to be understood as holding almost everywhere.

We first claim that for any $q \in Q_d$, the minimum principle

$$v \in H^1(I^d) \wedge [B_d(v, w, q) \geq 0 \text{ for all non-negative } w \in H^1(I^d)] \implies v \geq 0 \text{ in } I^d \tag{14}$$

holds (see [13], as well as [12], for an analogous inequality over smooth regions in $\mathbb{R}^d$). Indeed, suppose otherwise. Then the set $A = \{\, \mathbf{x} \in I^d : v(\mathbf{x}) < 0 \,\}$ has positive measure. Let

$$w(\mathbf{x}) = \max\{-v(\mathbf{x}), 0\} \qquad \forall\, \mathbf{x} \in \bar{I}^d.$$

By [21, Corollary 2.1.8], we have $w \in H^1(I^d)$. As in the proof of [20, Lemma 4.2], we easily see that $\nabla w = -\nabla v$ almost everywhere in I^d. Since $w \geq 0$ on $\bar{I}^d$, and $w = -v$ on A, we have

$$\begin{aligned} 0 \leq \int_{I^d} |\nabla v|^2 &= -\int_{I^d} \nabla v \cdot \nabla w = -B_d(v, w; q) + \langle qv, w\rangle_{L_2(I^d)} \\ &\leq \langle qv, w\rangle_{L_2(I^d)} = -\langle qv, v\rangle_{L_2(A)} \leq -q_0 \|v\|^2_{L_2(A)} < 0, \end{aligned}$$

a contradiction. Hence A cannot have positive measure, which establishes the claim.

Next, we claim that $S_d(0, g, \cdot)$ is antitone for any $g \geq 0$, i.e., that we have

$$[q_1, q_2 \in Q_d \text{ with } q_1 \geq q_2] \wedge [g \in L_2(\partial I^d) \text{ with } g \geq 0] \implies S_d(0, g, q_1) \leq S_d(0, g, q_2). \tag{15}$$

Indeed, let $v_i = S_d(0, g, q_i)$ for $i \in \{1, 2\}$, noting that $v_1 \geq 0$ by (14). Set $v = v_2 - v_1$. Now

$$B_d(v_1, \cdot; q_1) = 0 = B_d(v_2, \cdot; q_2) \qquad \text{on } H^1(I^d),$$

Thus for any non-negative $w \in H^1(I^d)$, we have

$$B_d(v, w; q_2) = \langle (q_1 - q_2)v_1, w\rangle_{L_2(I^d)} \geq 0.$$

Using (14), we see that $v \geq 0$, as claimed.

Now fix $f \in L_2(I^d)$, $g \in L_2(\partial I^d)$, and $q \in Q_d$. Let $u = S_d(0, g, q)$. Using [20, Lemma 4.3], we see that

$$\|S_d(f, g, q)\|_{L_\infty(I^d)} \leq \|S_d(f, \|g\|_{L_\infty(I^d)}, 0)\|_{L_\infty(I^d)} + \|u\|_{L_\infty(I^d)} \leq \frac{\|f\|_{L_\infty(I^d)}}{q_0} + \|u\|_{L_\infty(I^d)}.$$

Let $u_0 = S_d(0, \|g\|_{L_\infty(I^d)}, q_0)$. It is easy to check that

$$u_0(x) = \frac{\|g\|_{L_\infty(I^d)}}{\sqrt{q_0}\sinh\sqrt{q_0}} \sum_{j=1}^{d} \psi(x_j) \qquad \forall\, \mathbf{x} \in I^d,$$

where

$$\psi(t) = \cosh[\sqrt{q_0}t] + \cosh[\sqrt{q_0}(1-t)]$$

Since

$$\|u_0\|_{L_\infty(I^d)} = \eta_1 d \|g\|_{L_\infty(I^d)},$$

it only remains to show that

$$\|u\|_{L_\infty(I^d)} \leq \|u_0\|_{L_\infty(I^d)}. \tag{16}$$

Let $u_1 = S_d(0, \|g\|_{L_\infty(I^d)}, q)$, noting that $u_1 \geq 0$ by (14). Again using (14), we see that

$$B_d(u_1 - u, \cdot, q) = 0 \text{ on } H^1(I^d) \text{ and } u_1 - u = \|g\|_{L_\infty(I^d)} - g \geq 0 \text{ on } \partial I^d \implies u_1 \geq u \text{ in } I^d$$

and that

$$B_d(u + u_1, \cdot, q) = 0 \text{ on } H^1(I^d) \text{ and } u + u_1 = g + \|g\|_{L_\infty(I^d)} \geq 0 \text{ on } \partial I^d \implies u \geq -u_1 \text{ in } I^d,$$

and so

$$|u| \le u_1 \qquad \text{in } I^d.$$

Since $S_d(0, \|g\|_{L_\infty(I^d)}, \cdot)$ is antitone and $q \ge q_0$, we have

$$u_1 = S_d(0, \|g\|_{L_\infty(I^d)}, q) \le S_d(0, \|g\|_{L_\infty(I^d)}, q_0) = u_0 \qquad \text{in } I^d.$$

Combining these last two inequalities, we easily find that (16) holds, completing the proof of the Lemma. □

We now have the following perturbation estimate for the Neumann problem.

Theorem 3.2. *Let $[f, g, q], [\tilde{f}, \tilde{g}, \tilde{q}] \in L_2(I^d) \times L_2(\partial I^d) \times Q_d$. If we additionally know that $f \in L_\infty(I^d)$ and $g \in L_\infty(\partial I^d)$, then*

$$\begin{aligned}&\|S_d(f, g, q) - S_d(\tilde{f}, \tilde{g}, \tilde{q})\|_{H^1(I^d)}\\ &\quad\le \frac{1}{\min\{1, q_0\}}\Big[\|f - \tilde{f}\|_{L_2(I^d)} + (\eta_0\|f\|_{L_\infty(I^d)} + \eta_1 d\|g\|_{L_\infty(\partial I^d)})\|q - \tilde{q}\|_{L_2(I^d)} + \sqrt{2d+1}\|g - \tilde{g}\|_{L_2(\partial I^d)}\Big],\end{aligned}$$

where η_0 and η_1 are as in (13).

Proof. Let $u = S_d(f, g, q)$ and $\tilde{u} = S_d(\tilde{f}, \tilde{g}, \tilde{q})$. Setting $w = u - \tilde{u}$, we have

$$\langle f - \tilde{f}, w\rangle_{L_2(I^d)} + \langle g - \tilde{g}, w\rangle_{L_2(\partial I^d)} = B_d(u, w, q) - B_d(\tilde{u}, w, \tilde{q}) = B_d(w, w, q) + \langle (q - \tilde{q})u, w\rangle_{L_2(I^d)},$$

and so

$$\min\{1, q_0\}\|w\|^2_{H^1(I^d)} \le B_d(w, w, q) = \langle f - \tilde{f}, w\rangle_{L_2(I^d)} + \langle g - \tilde{g}, w\rangle_{L_2(\partial I^d)} - \langle (q - \tilde{q})u, w\rangle_{L_2(I^d)}. \tag{17}$$

Clearly

$$\left|\langle f - \tilde{f}, w\rangle_{L_2(I^d)}\right| \le \|f - \tilde{f}\|_{L_2(I^d)}\|w\|_{L_2(I^d)}. \tag{18}$$

Moreover, Lemma 3.1 yields

$$\begin{aligned}\left|\langle (q - \tilde{q})u, w\rangle_{L_2(I^d)}\right| &\le \|q - \tilde{q}\|_{L_2(I^d)}\|u\|_{L_\infty(I^d)}\|w\|_{L_2(I^d)}\\ &\le (\eta_0\|f\|_{L_\infty(I^d)} + \eta_1 d\|g\|_{L_\infty(\partial I^d)})\|q - \tilde{q}\|_{L_2(I^d)}\|w\|_{L_2(I^d)}.\end{aligned} \tag{19}$$

Finally, we may use Corollary 3.1 to see that

$$\left|\langle g - \tilde{g}, w\rangle_{L_2(\partial I^d)}\right| \le \|g - \tilde{g}\|_{L_2(\partial I^d)}\|w\|_{L_2(\partial I^d)} \le \sqrt{2d+1}\|g - \tilde{g}\|_{L_2(\partial I^d)}\|w\|_{H^1(I^d)}. \tag{20}$$

Substituting (18)–(20) into (17), the desired result follows. □

We now establish our desired error bound.

Corollary 3.2. *Let*

$$C_{d,\mathrm{Lip}} = d \cdot \frac{\max\left\{\sqrt{3}, (\eta_0 + \eta_1) \cdot \max\left\{\|\widetilde{\mathrm{App}}_{d-1,\infty}\|_{\mathrm{Lin}[H(\widetilde{K}_{d-1}), L_\infty(\partial I^d)]}, \|\,\mathrm{App}_{d,\infty}\,\|_{\mathrm{Lin}[H(K_d), L_\infty(I^d)]}\right\}\right\}}{\min\{1, q_0\}},$$

where η_0 and η_1 are as in (13). *If $[f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho})$ and $[\tilde{f}, \tilde{g}, \tilde{q}] \in H_d \times \widetilde{H}_{d-1} \times H_{d,\rho}$, then*

$$\left\|S_d(f, g, q) - S_d\big(\tilde{f}, \tilde{g}, \phi(\tilde{q})\big)\right\|_{H^1(I^d)} \le C_{d,\mathrm{Lip}}\left(\|f - \tilde{f}\|_{L_2(I^d)} + \|q - \tilde{q}\|_{L_2(I^d)} + \|g - \tilde{g}\|_{L_2(\partial I^d)}\right).$$

Proof. For $[f, g], [\tilde{f}, \tilde{g}] \in H_d \times \widetilde{H}_{d-1}$, we clearly have

$$\|f\|_{L_\infty(I^d)} \le \|\operatorname{App}_{d,\infty}\|_{\operatorname{Lin}[H(K_d),L_\infty(I^d)]} \qquad \text{and} \qquad \|g\|_{L_\infty(\partial I^d)} \le \|\widetilde{\operatorname{App}}_{d-1,\infty}\|_{\operatorname{Lin}[H(\widetilde{K}_{d-1}),L_\infty(\partial I^d)]}.$$

From the proof of [20, Lemma 4.5], we have

$$\|q - \phi(\tilde{q})\|_{L_2(I^d)} \le \|q - \tilde{q}\|_{L_2(I^d)}.$$

The desired result now follows from Theorem 3.2 and these inequalities, along with the fact that $d \in \mathbb{Z}^{++}$. □

The form we have chosen for the Lipschitz constant $C_{d,\mathrm{Lip}}$ was dictated by convenience. It is easy to see that Theorem 3.2 can give us a somewhat better value for $C_{d,\mathrm{Lip}}$, but the improvement is very slight.

Having established a value for the Lipschitz constant $C_{d,\mathrm{Lip}}$, we now establish bounds on the initial error. In what follows, we let $\operatorname{Int}_{d-1}\colon H(K_{d-1}) \to \mathbb{R}$ and $\operatorname{Int}_d\colon H(K_d) \to \mathbb{R}$ and denote the integration problems defined by

$$\operatorname{Int}_{d-1} z = \int_{I^{d-1}} z(\mathbf{y})\,d\mathbf{y} \qquad \forall\, z \in H(K_{d-1})$$

and

$$\operatorname{Int}_d z = \int_{I^d} z(\mathbf{y})\,d\mathbf{y} \qquad \forall\, z \in H(K_d),$$

respectively.

Theorem 3.3. *The initial error $e(0, S_d)$ satisfies the inequality*

$$\max\left\{\|\operatorname{Int}_d\|_{[H(K_d)]^*}, \frac{\sqrt{2d}\,\|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*}}{\max\{1, q_0\}}\right\} \le e(0, S_d)$$
$$\le \frac{\|\operatorname{App}_{d,2}\|_{\operatorname{Lin}[H(K_d),L_2(I^d)]} + \sqrt{2d+1}\,\|\operatorname{App}_{d-1,2}\|_{\operatorname{Lin}[H(K_{d-1}),L_2(I^{d-1})]}}{\min\{1, q_0\}}.$$

Proof. We first consider the lower bound. Since the relation $e(0, S_d) \ge \|\operatorname{Int}_d\|_{[H(K_d)]^*}$ was established in [20, Lemma 4.10], we only need to prove that

$$e(0, S_d) \ge \frac{\sqrt{2d}\,\|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*}}{\max\{1, q_0\}}.$$

Choose z in the unit ball of $H(K_{d-1})$ such that

$$\operatorname{Int}_{d-1} z = \|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*}.$$

Define $g\colon \partial I^d \to \mathbb{R}$ by

$$g(\mathbf{x}) = \frac{(-1)^\theta}{\sqrt{2d}} z(\hat{\mathbf{x}}_{[j]}) \qquad \text{if } x \in I^d_{j,\theta} \text{ for some } j \in \{1, \dots, d\}, \theta \in \{0, 1\}.$$

Then

$$\|g\|^2_{H(\widetilde{K}_{d-1})} = \sum_{\substack{1\le j\le d\\ \theta\in\{0,1\}}} \left\|g\big|_{I^d_{j,\theta}}\right\|^2_{H(K_{d-1})} = \frac{1}{2d}\sum_{\substack{1\le j\le d\\ \theta\in\{0,1\}}} \|z\|^2_{H(K_{d-1})} = \|z\|^2_{H(K_{d-1})} = 1,$$

and so $g \in \widetilde{H}_{d-1}$. Now let $u = S_d(0, g; q_0)$. Since $\|1\|_{H^1(I^d)} = 1$, we have

$$\max\{1, q_0\}\|u\|_{H^1(I^d)} \ge B_d(u, 1; q_0) = \langle g, 1\rangle_{L_2(\partial I^d)} = \int_{\partial I^d} g(\mathbf{x})\,d\mathbf{x} = \sum_{\substack{1\le j\le d\\ \theta\in\{0,1\}}} \int_{I^d_{j,\theta}} (-1)^\theta g(\mathbf{x})\,d\mathbf{x}$$
$$= 2d \cdot \frac{1}{\sqrt{2d}} \cdot \int_{I^{d-1}} z(\mathbf{y})\,d\mathbf{y} = \sqrt{2d}\int_{I^{d-1}} z(\mathbf{y})\,d\mathbf{y} = \sqrt{2d}\,\|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*}.$$

Hence

$$e(0, S_d) \geq \|u\|_{H^1(I^d)} \geq \frac{\sqrt{2d}\, \|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*}}{\max\{1, q_0\}},$$

as required.

We now turn to the upper bound. Choose $[f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (H_{d,\rho} \cap Q_d)$, and let $u = S_d(f, g, q)$. From (5), we see that

$$B_d(u, u; q) = \langle f, u\rangle_{L_2(I^d)} + \langle g, u\rangle_{L_2(\partial I^d)}.$$

By the definition of B_d and Corollary 3.1, we see that

$$\begin{aligned}\min\{1, q_0\}\|u\|^2_{H^1(I^d)} &\leq \|f\|_{L_2(I^d)}\|u\|_{L_2(I^d)} + \|g\|_{L_2(\partial I^d)}\|w\|_{L_2(\partial I^d)} \\ &\leq [\|f\|_{L_2(I^d)} + \sqrt{2d+1}\,\|g\|_{L_2(\partial I^d)}]\|u\|_{H^1(I^d)},\end{aligned}$$

and so

$$\begin{aligned}\|S(f, g, q)\|_{H^1(I^d)} = \|u\|_{H^1(I^d)} &\leq \frac{\|f\|_{L_2(I^d)} + \sqrt{2d+1}\,\|g\|_{L_2(\partial I^d)}}{\min\{1, q_0\}} \\ &\leq \frac{\|\operatorname{App}_{d,2}\|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} + \sqrt{2d+1}\,\|\operatorname{App}_{d-1,2}\|_{\operatorname{Lin}[H(K_{d-1}), L_2(I^{d-1})]}}{\min\{1, q_0\}}.\end{aligned}$$

Since $[f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (H_{d,\rho} \cap Q_d)$ is arbitrary, the result follows immediately. □

Noting the presence of the $\Theta(\sqrt{d})$-factor on both the upper and lower bounds on $e(0, S_d)$, we see that these bounds are fairly tight, assuming that the norms involved are reasonably close to each other.

4 Reduction to the L_2-approximation problem

Suppose that we know how to do L_2-approximation of functions from $H(K_d)$ and $H(K_{d-1})$. From the latter, it follows that we can do L_2-approximation for $H(\widetilde{K}_{d-1})$. We can then approximate $S_d(f, g, q)$ by $S_d(\tilde{f}, \tilde{g}, \tilde{q})$, where the tildes denote approximations. Using Corollary 3.2, we can then estimate the error in this approximation, which allows us to obtain bounds on the ε-cardinality number.

Let us make this more precise. First, let $W_d = (\operatorname{App}_{d,2})^*(\operatorname{App}_{d,2})$, which is a compact self-adjoint operator on $H(K_d)$ having finite trace. Let $\{(\lambda_{d,j}, e_{d,j})\}_{j\in\mathbb{Z}^{++}}$ denote the eigenpairs of W_d, with $\{e_{d,j}\}_{j\in\mathbb{Z}^{++}}$ being an orthonormal basis for $H(K_d)$ and with

$$\lambda_{d,1} \geq \lambda_{d,2} \geq \ldots 0 \qquad \text{and} \qquad \lim_{j\to\infty} \lambda_{d,j} = 0.$$

As in [17], we consider the problem $\operatorname{App}_{d,2}$ of approximating functions from H_d in the L_2-norm. We first consider Λ^{all}. Using results from [15, Chapter 4], we find that the algorithm $A_{d,n,\text{all}}$ defined by

$$A_{d,n,\text{all}} f = \sum_{j=1}^{n} \langle f, e_{d,j,}\rangle_{H(K_d)} e_{d,j} \qquad \forall\, f \in H_d$$

is an nth minimal error algorithm using Λ^{all}, and that the nth minimal error satisfies

$$r(n, \operatorname{App}_{d,2}, \Lambda^{\text{all}}) = e(A_{d,n,\text{all}}, \operatorname{App}_{d,2}, \Lambda^{\text{all}}) = \sqrt{\lambda_{d,n+1}}.$$

Moreover, since

$$\sum_{j=1}^{\infty} \lambda_{d,j} = \operatorname{trace} W_d = \int_{I^d} K_d(\mathbf{x}, \mathbf{x})\, d\mathbf{x} < \infty,$$

we see that

$$r(n, \mathrm{App}_{d,2}, \Lambda^{\mathrm{all}}) \le \frac{\sqrt{\mathrm{trace}\, W_d}}{\sqrt{n+1}}.$$

Letting

$$r_{d,\mathrm{all}} = \sup\left\{ r \ge 0 : \lim_{j\to\infty} \lambda_{d,j} n^r = 0 \right\},$$

we see that $r_{d,\mathrm{all}} \ge \frac{1}{2}$. Note that this result holds for any kernel K_d satisfying (6). For smoother kernels, the eigenvalues of W_d might decay more quickly to zero, which would give a larger value for $r_{d,\mathrm{all}}$.

Next, we consider Λ^{std}. Again following [17], we can use the result for Λ^{all}, along with the fact that

$$r(n, \mathrm{App}_{d,2}, \Lambda^{\mathrm{std}}) \le \min_{j\in\mathbb{Z}^+} \left(r^2(j, \mathrm{App}_{d,2}, \Lambda^{\mathrm{all}}) + \frac{\mathrm{trace}\, W_d}{n} \right)^{1/2},$$

to see that

$$r(n, \mathrm{App}_{d,2}, \Lambda^{\mathrm{std}}) \le \frac{\sqrt{2\,\mathrm{trace}\, W_d}}{n^{1/4}}.$$

More recently, Kuo et al. [9] have shown that

$$\frac{2r_{d,\mathrm{all}}}{2r_{d,\mathrm{all}}+1} r_{d,\mathrm{all}} \le r_{d,\mathrm{std}} \le r_{d,\mathrm{all}}.$$

Furthermore, they have conjectured that if (6) holds, then $r_{d,\mathrm{all}} = r_{d,\mathrm{std}}$.

We summarize these observations in

Lemma 4.1. *Let $k \in \{\mathrm{all}, \mathrm{std}\}$. There exist $r_{d,\mathrm{all}} \ge \frac{1}{2}$ and $r_{d,\mathrm{std}} \ge \frac{1}{4}$, as well as $C_{d,k} > 0$, such that there is a linear algorithm $A_{d,n,k}$ for the approximation problem using n evaluations from Λ^k, whose error satisfies*

$$e(A_{d,n,k}, \mathrm{App}_{d,2}, \Lambda^k) \le \frac{C_{d,k}}{n^{r_{d,k}}}.$$

For Λ^{all}, we may take

$$r_{d,\mathrm{all}} = \tfrac{1}{2} \qquad \textit{and} \qquad C_{d,\mathrm{all}} = \sqrt{\mathrm{trace}\, W_d},$$

whereas for Λ^{std}, we may take

$$r_{d,\mathrm{std}} = \tfrac{1}{4} \qquad \textit{and} \qquad C_{d,\mathrm{std}} = \sqrt{2\,\mathrm{trace}\, W_d}. \quad \square$$

We now are ready to prove a result that tells us that

$$\mathrm{card}(\varepsilon, S_d, \Lambda^k) = O\left(\max\{\mathrm{card}(\varepsilon, \mathrm{App}_{d,2}, \Lambda^k), \mathrm{card}(\varepsilon, \mathrm{App}_{d-1,2}, \Lambda^k)\} \right).$$

Theorem 4.1. *The Neumann problem is no harder than the L_2-approximation problem. More precisely, for $k \in \{\mathrm{all}, \mathrm{std}\}$, we find that the following hold:*

1. *For any $n \in \mathbb{Z}^+$, there exists a linear algorithm $U_{d,n,k}$ using n evaluations from Λ^k, such that*

$$e(U_{d,n,k}, S_d, \Lambda^k) \le \frac{C^*_{d,k}}{n^{r^*_{d,k}}}, \tag{21}$$

where (using the notation of Lemma 4.1)

$$C^*_{d,k} = C_{d,\mathrm{Lip}}[3^{r_{d,k}} C_{d,k}(\rho+1) + (6d)^{r_{d-1,k}} C_{d-1,k}]$$

and

$$r^*_{d,k} = \min\{r_{d,k}, r_{d-1,k}\}.$$

Hence,

$$\mathrm{card}(\varepsilon, S_d, \Lambda^k) \le \left\lceil \left(\frac{C^*_{d,k}}{\varepsilon\, \mathrm{ErrCrit}(S_d)} \right)^{1/r^*_{d,k}} \right\rceil.$$

2. *Let*

$$r_k = \inf_{\ell \in \mathbb{Z}^{++}} r_{\ell,k}, \tag{22}$$

noting that we can always find algorithms such that

$$r_{\text{all}} \geq \tfrac{1}{2} \qquad \textit{and} \qquad r_{\text{std}} \geq \tfrac{1}{4}.$$

Then

$$\text{card}(\varepsilon, S_d, \Lambda^k) \leq \left\lceil \left(\frac{C^*_{d,k}}{\varepsilon \, \text{ErrCrit}(S_d)} \right)^{1/r_k} \right\rceil,$$

with

$$C^*_{d,k} = 6^{r_k} (\rho + 1) \, d^{r_k} \, C_{d,\text{Lip}} (C_{d,k} + C_{d-1,k}). \tag{23}$$

Proof. Let $k \in \{\text{all}, \text{std}\}$. For $n \in \mathbb{Z}^+$, define an algorithm $A^*_{d-1,n,k}$ for $L_2(\partial I^d)$-approximation of functions from $H(\widetilde{K}_{d-1})$ by

$$\left(A^*_{d-1,n,k} \, g\right)\big|_{I^d_{j,\theta}} = A_{d-1,\lfloor n/(2d) \rfloor,k} \left(g\big|_{I^d_{j,\theta}} \right) \qquad \forall \, g \in H(\widetilde{K}_{d-1}), \, j \in \{1, \ldots, d\}, \, \theta \in \{0, 1\}.$$

We clearly have

$$\|g - A^*_{d-1,n,k} g\|_{L_2(\partial I^d)} \leq \frac{C_{d-1,k}}{\lfloor n/(2d) \rfloor^{r_{d-1,k}}} \|g\|_{H(\widetilde{K}_{d-1})} \qquad \forall \, g \in H(\widetilde{K}_{d-1}).$$

Now let $[f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho})$. Pick $n \in \mathbb{Z}^+$; we assume that $(6d)|n$ without loss of generality. We then find that

$$U_{d,n,k}(f, g, q) = S_d \left(A_{d,n/3,k} f, \widetilde{A}_{d-1,n/3,k} g, \phi(A_{d,n/3} q) \right).$$

From Corollary 3.2, we see that

$$\begin{aligned}
&\|S_d(f, g, q) - U_{d,n,k}(f, g, q)\|_{H(I^d)} \\
&\qquad \leq C_{d,\text{Lip}} [\|f - A_{d,n/3,k} f\|_{L_2(I^d)} + \|q - A_{d,n/3,k} q\|_{L_2(I^d)} + \|g - A^*_{d-1,n/3,k} g\|_{L_2(\partial I^d)}] \\
&\qquad \leq C_{d,\text{Lip}} \left[\frac{C_{d,k}(\|f\|_{H(K_d)} + \|q\|_{H(K_d)})}{(n/3)^{r_{d,k}}} + \frac{C_{d-1,k} \|g\|_{H(\widetilde{K}_{d-1})}}{[n/(6d)]^{r_{d-1,k}}} \right] \\
&\qquad \leq \frac{C^*_{d,k}}{n^{r^*_{d,k}}},
\end{aligned}$$

which implies that (21) holds. The remainder of the theorem follows immediately. □

5 Notions of tractability

So far, we have treated the number of variables d as a fixed parameter. In the remainder of this paper, we consider a sequence $S = \{S_d\}_{d \in \mathbb{Z}^{++}}$ of Neumann problems, studying the dependence of $\text{card}(\varepsilon, S_d, \Lambda)$ on both ε and d.

First, we describe various levels of tractability for our Neumann problem, see (e.g.) [10] for discussion. The problem S is said to be *weakly tractable* in the class Λ if

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln \text{card}(\varepsilon, S_d, \Lambda)}{\varepsilon^{-1} + d} = 0.$$

A problem is weakly tractable iff the cardinality number grows subexponentially in ε^{-1} and d. The problem S is said to be (polynomially) *tractable* in the class Λ if there exist non-negative numbers C, p_{err}, and p_{dim} such that

$$\text{card}(\varepsilon, S_d, \Lambda) \leq C \left(\frac{1}{\varepsilon} \right)^{p_{\text{err}}} d^{p_{\text{dim}}} \qquad \forall \, \varepsilon \in (0, 1), \, d \in \mathbb{Z}^{++}. \tag{24}$$

Numbers $p_{\rm err} = p_{\rm err}(\Lambda)$ and $p_{\rm dim} = p_{\rm dim}(\Lambda)$ such that (24) holds are called ε- and *d-exponents of tractability*; these need not be uniquely defined. Finally, the problem S is said to be *strongly* (polynomially) *tractable* in the class Λ if $p_{\rm dim} = 0$ in (24); in this case, we define

$$p_{\rm strong}(\Lambda) = \inf \left\{ p_{\rm err} \ge 0 : \exists\, C \ge 0 \text{ such that } \operatorname{card}(\varepsilon, S_d, \Lambda) \le C \left(\frac{1}{\varepsilon}\right)^{p_{\rm err}} \forall\, \varepsilon \in (0,1),\, d \in \mathbb{Z}^{++} \right\}$$

to be the *exponent of strong tractability*.

Using Theorem 4.1, we immediately have

Corollary 5.1. *Let $k \in \{\text{all}, \text{std}\}$. Suppose that* (22) *holds. Using the notation of Lemma* 4.1*, the following hold:*

1. *If*
$$\lim_{d\to\infty} \frac{1}{d} \ln \frac{C_{d,\rm Lip}(C_{d,k} + C_{d-1,k})}{\operatorname{ErrCrit}(S_d)} = 0, \tag{25}$$
then the Neumann problem is weakly tractable.

2. *If there exists $\pi_k > 0$ such that*
$$\frac{C^*_{d,k}}{\operatorname{ErrCrit}(S_d)} = O(d^{\pi_k}),$$
then the Neumann problem is tractable, with
$$p_{\rm err}(\Lambda^k) \le \frac{1}{r_k} \qquad \textit{and} \qquad p_{\rm dim}(\Lambda^k) \le \frac{\pi_k}{r_k}\,.$$

3. *If*
$$\limsup_{d\to\infty} \frac{C^*_{d,k}}{\operatorname{ErrCrit}(S_d)} < \infty,$$
then the Neumann problem is strongly tractable, with
$$p_{\rm strong}(\Lambda^k) \le \frac{1}{r_k}\,. \quad \square$$

Of course, a problem's weak tractability, tractability, or strong tractability will depend on the error criterion used. Hence we let $p^{\rm abs}_{\rm err}$, $p^{\rm abs}_{\rm dim}$, and $p^{\rm abs}_{\rm strong}$ denote the ε- and d-exponents of tractability and the exponent of strong tractability under the absolute error criterion. When we are using the normalized error criterion, we shall denote these exponents by $p^{\rm nor}_{\rm err}$, $p^{\rm nor}_{\rm dim}$, and $p^{\rm nor}_{\rm strong}$.

6 Weighted reproducing kernel Hilbert spaces

Up until this point, we have assumed very little about the reproducing kernel K_d. Other than condition (6), the kernels K_d can be arbitrary. If we want to study tractability, we will need to say something about how the kernels K_d are related for $d \in \mathbb{Z}^{++}$. The standard approach is to use weighted kernels.

Let $H(K)$ be a separable RKHS of functions defined over $\bar{I}$, where the "master" reproducing kernel K is a measurable non-zero function defined on $\bar{I} \times \bar{I}$. We will require that $K \in L_\infty(I \times I)$, so that

$$\kappa_0 := \operatorname*{ess\,sup}_{t\in I} K(t,t) < \infty. \tag{26}$$

It then follows that

$$0 \le \kappa_2 \le \kappa_1 \le \kappa_0,$$

where

$$\kappa_1 = \int_0^1 K(t,t)\,dt \tag{27}$$

and

$$\kappa_2 = \int_0^1 \int_0^1 K(s,t)\,dt\,ds. \tag{28}$$

Note that κ_0 and κ_1 are positive, but that κ_2 may be equal to zero. As we shall see, tractability results will be different for the cases $\kappa_2 > 0$ and $\kappa_2 = 0$.

Example. The Korobov and min kernels are defined as follows:

1. Let $r \in \mathbb{Z}^{++}$. The rth *Korobov kernel* $K_{\mathrm{Kor},r}$ is defined as

$$K_{\mathrm{Kor},r}(s,t) = \frac{(-1)^{r+1} B_{2r}(\{s-t\})}{(2r)!} \qquad \forall\, s,t \in [0,1], \tag{29}$$

   where B_{2r} is the Bernoulli polynomial of order $2r$ and $\{s\}$ denotes the fractional part of $s \in \mathbb{R}$. For $K_{\mathrm{Kor},r}$, we find $\kappa_0 = \kappa_1 = B_{2r}/(2r)!$ and $\kappa_2 = 0$, where B_{2r} is the $2r$th Bernoulli number.

2. The *min kernel* $K_{\min}$ is defined as

$$K_{\min}(s,t) = \min\{s,t\} \qquad \forall\, s,t \in [0,1]. \tag{30}$$

   For $K_{\min}$, we find that $\kappa_0 = 1$, $\kappa_1 = \frac{1}{2}$, and $\kappa_2 = \frac{1}{3}$.

These kernels have been extensively studied in many papers, see [10] and the references cited therein. □

We let $|\mathfrak{u}|$ denote the size of $\mathfrak{u} \subseteq [d]$, where $[d] = \{1,\ldots,d\}$. Let

$$\boldsymbol{\gamma} = \{\, \gamma_{d,\mathfrak{u}} : \mathfrak{u} \subseteq [d],\, d \in \mathbb{Z}^{++} \,\}$$

be a set of non-negative *weights* $\gamma_{d,\mathfrak{u}}$.

Example. What kinds of weights have been most thoroughly studied?

1. *Product weights* [14]. Here,

$$\gamma_{d,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{d,j} \qquad \text{with} \qquad \gamma_{d,1} \ge \gamma_{d,2} \ge \cdots \ge \gamma_{d,d} \ge 0. \tag{31}$$

2. *Finite-order weights* [4] of *order* $\omega \in \mathbb{Z}^{++}$. Here

$$\gamma_{d,\mathfrak{u}} \ne 0 \quad \text{only if} \quad |\mathfrak{u}| \le \omega \qquad \forall\, \mathfrak{u} \subseteq [d],\, d \in \mathbb{Z}^{++}, \tag{32}$$

   where ω is the smallest positive integer such that (32).

Other classes of weights (such as finite-diameter weights and order-dependent weights) have been studied as well; again, see [10]. □

For each $d \in \mathbb{Z}^{++}$, the space $H(K_d)$ will be the RKHS whose reproducing kernel is

$$K_d = \sum_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} K_{d,\mathfrak{u}},$$

with

$$K_{d,\mathfrak{u}}(\mathbf{x},\mathbf{y}) = \prod_{j\in\mathfrak{u}} K(x_j, y_j) \qquad \forall\, \mathbf{x},\mathbf{y} \in \bar{I}^d,\, \mathfrak{u} \subseteq [d].$$

Recall that we require $q_0 \in H_{d,\rho}$ for all $d \in \mathbb{Z}^{++}$. It is known (see, e.g., [17]) that if $\gamma_{d,\emptyset} > 0$, then the constant function 1 belongs to $H(K_d)$, with $\|1\|_{H(K_d)} \le \gamma_{d,\emptyset}^{-1/2}$. Hence we need $q_0 \gamma_{d,\emptyset}^{-1/2} \le \rho$ to hold for all d. Since q_0 and ρ are to be independent of d, this latter condition can hold iff

$$\gamma_{\min,\emptyset} := \inf_{d\in\mathbb{Z}^{++}} \gamma_{d,\emptyset} > 0. \tag{33}$$

Hence we shall assume that condition (33) holds in the rest of this paper, and that q_0 and ρ satisfy

$$q_0 \le \gamma_{\min,\emptyset}^{1/2}\,\rho. \tag{34}$$

Since (33) and (34) both hold, we now know that $q_0 \in H_{d,\rho}$ for all $d \in \mathbb{Z}^{++}$.

Remark 6.1. The intuition behind the definition of $H(K_d)$ is that any function belonging to this space can be decomposed as a sum of simpler functions. Let us make this precise, under the simplifying assumption that $1 \notin H(K)$, which happens iff $\kappa_2 > 0$ (see, e.g., [17, Lemma 1]). For $\mathfrak{u} \subseteq [d]$, let $H(K_{d,\mathfrak{u}})$ denote the RKHS whose reproducing kernel is $K_{d,\mathfrak{u}}$, noting that a function belonging to $H(K_{d,\mathfrak{u}})$ depends only on the variables x_i for $i \in \mathfrak{u}$. Then for any $f \in H(K_d)$, there is a unique decomposition of the form

$$f = \sum_{\mathfrak{u}\subseteq[d]} f_{\mathfrak{u}},$$

with $f_{\mathfrak{u}} \in H(K_{d,\mathfrak{u}})$ for all $\mathfrak{u} \subseteq [d]$, with

$$\|f\|_{H(K_d)}^2 = \sum_{\mathfrak{u}\subseteq[d]} \frac{1}{\gamma_{d,\mathfrak{u}}} \|f_{\mathfrak{u}}\|_{H(K_{d,\mathfrak{u}})}^2,$$

once again see [17]. Note that for this last sum to be finite, then $f_{\mathfrak{u}}$ must be zero whenever $\gamma_{d,\mathfrak{u}} = 0$, and we must use the notational convention that $0/0 = 0$. Thus a function belonging to a weighted RKHS can be written as a sum of simpler functions. In particular, note that for finite-order weights of order ω, we have

$$f = \sum_{\substack{\mathfrak{u}\subseteq[d]\\ |\mathfrak{u}|\le\omega}} f_{\mathfrak{u}}$$

with

$$\|f\|_{H(K_d)}^2 = \sum_{\substack{\mathfrak{u}\subseteq[d]\\ |\mathfrak{u}|\le\omega}} \frac{1}{\gamma_{d,\mathfrak{u}}} \|f_{\mathfrak{u}}\|_{H(K_{d,\mathfrak{u}})}^2.$$

Thus a function belonging to a weighted RKHS based on finite-order weight of order ω can be written as a sum of simpler functions, with each term depending on at most ω variables. □

For $\ell \in \mathbb{Z}^{++}$, let us define $\sigma_\ell \colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\sigma_\ell(\theta) = \left(\sum_{\mathfrak{u}\in[\ell]} \gamma_{d,\mathfrak{u}}\,\theta^{|\mathfrak{u}|}\right)^{1/2} \qquad \forall\,\theta \in \mathbb{R}^+.$$

We can use the functions σ_{d-1} and σ_d to estimate various norms, as well as the Lipschitz constant for our Neumann problem. Before doing this, it will be useful to let $W = (\mathrm{App})^*(\mathrm{App}) \in \mathrm{Lin}[H(K)]$, where $\mathrm{App} \in \mathrm{Lin}[H(K), L_2(I)]$ is the embedding operator. More explicitly,

$$Wf = \int_0^1 K(\cdot, y) f(y)\,dy \qquad \forall f \in H(K).$$

Lemma 6.1. *The following estimates hold for weighted RKHSs:*

1. $\sigma_d(\kappa_2) \le \|\,\mathrm{App}_{d,2}\,\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} \le \sigma_d(\kappa_1)$.
2. *If* $\kappa_2 = 0$, *then*
$$\|\,\mathrm{App}_{d,2}\,\|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} = \max_{\mathfrak{u}\subseteq[d]} \left[\gamma_{d,\mathfrak{u}} \|W\|_{\mathrm{Lin}[H(K)]}^{\mathfrak{u}}\right]^{1/2}.$$
3. $\sigma_{d-1}(\kappa_2) \le \|\widetilde{\mathrm{App}}_{d-1,2}\|_{\mathrm{Lin}[H(\widetilde{K}_{d-1}),L_2(\partial I^d)]} \le \sigma_{d-1}(\kappa_1)$.

4. $\| \mathrm{App}_{d,\infty} \|_{\mathrm{Lin}[H(K_d),L_\infty(I^d)]} \le \sigma_d(\kappa_0)$.

5. $\|\widetilde{\mathrm{App}}_{d-1,\infty}\|_{\mathrm{Lin}[H(\widetilde{K}_{d-1}),L_\infty(\partial I^d)]} \le \sigma_{d-1}(\kappa_0)$.

6. $\| \mathrm{Int}_d \|_{[H(K_d)]^*} = \sigma_d(\kappa_2)$.

7. $\| \mathrm{Int}_{d-1} \|_{[H(K_{d-1})]^*} = \sigma_{d-1}(\kappa_2)$.

8. *Let*

$$\eta_2 = \frac{\max\left\{\eta_0 + \eta_1, \sqrt{3/\gamma_{\min,\emptyset}}\right\}}{\min\{1, q_0\}}, \tag{35}$$

where η_0 and η_1 are as in (13)*. The Lipschitz constant $C_{d,\mathrm{Lip}}$ for our Neumann problem satisfies*

$$C_{d,\mathrm{Lip}} \le \eta_2 d \ \max\{\sigma_d(\kappa_0), \sigma_{d-1}(\kappa_0)\}. \tag{36}$$

9. *The initial error satisfies the inequality*

$$\max\left\{\sigma_d(\kappa_2), \frac{\sqrt{2d}\,\sigma_{d-1}(\kappa_2)}{\max\{1, q_0\}}\right\} \le e(0, S_d) \le \frac{\sigma_d(\kappa_1) + \sqrt{2d+1}\,\sigma_{d-1}(\kappa_1)}{\min\{1, q_0\}}.$$

Proof. The norm estimates may be found in [17].

Using the bounds on $\| \mathrm{App}_{d,\infty} \|_{\mathrm{Lin}[H(K_d),L_\infty(I^d)]}$ and $\|\widetilde{\mathrm{App}}_{d-1,\infty}\|_{\mathrm{Lin}[H(\widetilde{K}_{d-1}),L_\infty(\partial I^d)]}$ along with the fact that $\sigma_l(\theta) \ge \gamma_{\min,\emptyset}$ for any $l \in \mathbb{Z}^{++}$ and $\theta \in \mathbb{R}^+$, we get the bound on $C_{d,\mathrm{Lip}}$.

The bounds on the initial error follows from Theorem 3.3 and the bounds on the various operator norms found in the rest of the Lemma. □

7 Tractability results for product and finite-order weights

Suppose that K is any reproducing kernel satisfying (26). What can we say about the tractability of our Neumann problem? We will be especially interested in knowing whether tractability of L_2-approximation implies tractability of our Neumann problem. We remind the reader that the L_2-approximation problem for problem elements H_d and information Λ^k (where $k \in \{\mathrm{all}, \mathrm{std}\}$) is tractable iff

$$\frac{e(n, \mathrm{App}_d, \Lambda^k)}{\mathrm{ErrCrit}(\mathrm{App}_d)} \le \frac{C_{k,\mathrm{ErrCrit}} d^{s_{k,\mathrm{ErrCrit}}}}{n^{r_k}}$$

for any $d \in \mathbb{Z}^{++}$ and $n \in \mathbb{Z}^+$. Here, $C_{k,\mathrm{ErrCrit}} > 0$, $r_{k,\mathrm{ErrCrit}} > 0$, and $s_{k,\mathrm{ErrCrit}} \ge 0$ are independent of d and n. Moreover, the error criterion for approximation is defined to be

$$\mathrm{ErrCrit}(\mathrm{App}_{d,2}) = \begin{cases} 1 & \text{for absolute error,} \\ e(0, \mathrm{App}_{d,2}) = \| \mathrm{App}_{d,2} \|_{\mathrm{Lin}[H(K_d),L_2(I^d)]} & \text{for normalized error,} \end{cases}$$

which is analogous to (12) for the Neumann problem.

Note that L_2-approximation is tractable iff we may take

$$r_{d,k} = r_k \qquad \text{and} \qquad C_{d,k} = C_{k,\mathrm{ErrCrit}}\, \mathrm{ErrCrit}(\mathrm{App}_d)\, d^{s_{k,\mathrm{ErrCrit}}} \tag{37}$$

in Lemma 4.1. The following simple lemma will be helpful in establishing tractability of the Neumann problem.

Lemma 7.1. *If L_2-approximation is tractable, then the Neumann problem is tractable. More precisely, let $k \in \{\mathrm{all}, \mathrm{std}\}$ and suppose that the following hold:*

1. *L_2-approximation is tractable, so that* (37) *holds in Lemma* 4.1.

2. *There exists $C_{\mathrm{Lip}} > 0$ and $t_{\mathrm{Lip}} \geq 0$ such that the Lipschitz constant $C_{\mathrm{Lip},d}$ for the Neumann problem is bounded by*

$$C_{d,\mathrm{Lip}} \leq C_{\mathrm{Lip}} d^{t_{\mathrm{Lip}}}$$

for all $d \in \mathbb{Z}^{++}$.

3. *There exists $C_{\mathrm{ErrCrit}} > 0$ and $u_{\mathrm{ErrCrit}} \geq 0$ such that*

$$\frac{\max\{\mathrm{ErrCrit}(\mathrm{App}_{d,2}), \mathrm{ErrCrit}(\mathrm{App}_{d-1,2})\}}{\mathrm{ErrCrit}(S_d)} \leq C_{\mathrm{ErrCrit}} d^{u_{\mathrm{ErrCrit}}}$$

for all $d \in \mathbb{Z}^{++}$.

Then the Neumann problem is tractable, with

$$p_{\mathrm{err}}(\Lambda^k) = \frac{1}{r_k} \qquad \text{and} \qquad p_{\mathrm{dim}}(\Lambda^k) = 1 + \frac{s_{k,\mathrm{ErrCrit}} + t_{\mathrm{Lip}} + u_{\mathrm{ErrCrit}}}{r_k}.$$

Proof. This follows immediately from (23) and Corollary 5.1. □

7.1 Results for product weights

In this subsection, we give tractability results for product weights

$$\gamma_{d,\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_{d,j}, \qquad \forall\, d \in \mathbb{Z}^{++},$$

where

$$\gamma_{d,1} \geq \gamma_{d,2} \geq \cdots \geq \gamma_{d,d} \geq 0 \qquad \forall\, d \in \mathbb{Z}^{++}.$$

We first consider weak tractability. Let us say that the sum of the weights is *sublinearly bounded* if

$$\lim_{d \to \infty} \frac{1}{d} \sum_{j=1}^{d} \gamma_{d,j} = 0. \tag{38}$$

Then we have the following result.

Theorem 7.1. *Sublinear boundedness is essentially necessary and sufficient for weak tractability. More precisely:*

1. *If the sum of the weights is sublinearly bounded, then the Neumann problems $(S_d, \Lambda^{\mathrm{all}})$ and $(S_d, \Lambda^{\mathrm{std}})$ are weakly tractable under both the absolute and normalized error criteria.*

2. *Suppose that the sum of the weights is not sublinearly bounded. Then there exists a kernel K such that the Neumann problem $(S_d, \Lambda^{\mathrm{std}})$ is not weakly tractable under the absolute error criterion.*

Proof. For the first part, note that since $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$, we only need to show that the problem is weakly tractable for Λ^{std}. Also, note that since we are using product weights, Lemma 6.1 implies that $e(0, S_d) \geq \sigma_d(\kappa_2) \geq 1$. So, it suffices to establish weak tractability for standard information under the absolute error criterion. We use Corollary 5.1, noting that (22) holds with $r_{\mathrm{std}} = \frac{1}{4}$ and

$$C_{\ell,\mathrm{std}} \leq \sqrt{2}\, \sigma_\ell(\kappa_1) \qquad \text{for } \ell \in \mathbb{Z}^{++}.$$

Using (36), we find that

$$\begin{aligned} C_{d,\mathrm{Lip}}(C_{d,\mathrm{std}} + C_{d-1,\mathrm{std}}) &\leq \eta_2 d \, \max\{\sigma_d(\kappa_0), \sigma_{d-1}(\kappa_0)\} \cdot \sqrt{2}[\sigma_d(\kappa_1) + \sigma_{d-1}(\kappa_1)] \\ &\leq 2\eta_2 \sqrt{2}\, d \max\{\sigma_d(\kappa_0), \sigma_{d-1}(\kappa_0)\}. \end{aligned}$$

For $\ell \in \mathbb{Z}^{++}$, we have

$$\sigma_\ell(\theta) = \prod_{j=1}^{\ell} (1 + \theta\gamma_{\ell,j})^{1/2},$$

and so

$$\ln \sigma_\ell(\theta) = \tfrac{1}{2} \sum_{j=1}^{\ell} \ln(1 + \theta\gamma_{\ell,j}) \le \tfrac{1}{2} \left(\sum_{j=1}^{\ell} \gamma_{\ell,j} \right) \theta = \hat{\sigma}_\ell \theta,$$

where

$$\hat{\sigma}_\ell = \tfrac{1}{2} \sum_{j=1}^{\ell} \gamma_{\ell,j}.$$

Since we are using product weights, we have $\sigma_\ell(\kappa_1) \ge 1$ for $\ell \in \mathbb{Z}^{++}$, and so

$$\ln[C_{d,\mathrm{Lip}}(C_{d,\mathrm{std}} + C_{d-1,\mathrm{std}})] \le \ln(2\eta_2\sqrt{2}) + \ln d + (\hat{\sigma}_d + \hat{\sigma}_{d-1})\kappa_0.$$

Since (38) holds, we find that (25) holds, and so the problem is weakly tractable.

To show the second part of the theorem, consider the kernel $K \colon \bar{I} \times \bar{I} \to \mathbb{R}$ as

$$K(x, y) = \tfrac{1}{2} \left(\left| x - \tfrac{1}{2} \right| + \left| y - \tfrac{1}{2} \right| + |x - y| \right) \qquad \forall\, x, y \in \bar{I}.$$

Consider the integration problem $\mathrm{Int}_d \colon H(K_d) \to \mathbb{R}$ defined by

$$\mathrm{Int}_d\, z = \int_{I^d} z(\mathbf{x})\, d\mathbf{x} \qquad \forall\, z \in H(K_d),$$

noting that

$$\| \mathrm{Int}_d \|_{[H(K_d)]^*} = \sigma_d(\kappa_2) \ge 1.$$

From [5, Theorem 7.1], we know that for this kernel K, the problem $(\mathrm{Int}_d, \Lambda^{\mathrm{std}})$ is not weakly tractable under the normalized error criterion, since (38) does not hold. We claim that Int_d can be reduced to $S_d(\cdot, 0; 1)$. To see this, let $f \in H_d$, and let $u = S_d(f, 0, ; 1)$, so that

$$\mathrm{Int}_d\, f = \langle f, 1 \rangle_{L_2(I^d)} = B_d(u, 1; 1) = \mathrm{Int}_d\, u. \tag{39}$$

For $\varepsilon > 0$, compute an approximation $\tilde{u}_\varepsilon$ of u such that $\|u - \tilde{u}_\varepsilon\|_{H^1(I^d)} \le \varepsilon$, using $\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, \Lambda^{\mathrm{std}})$ evaluations of f. Now define $\mathrm{Int}_{d,\varepsilon}\, f = \mathrm{Int}_d\, \tilde{u}_\varepsilon$, noting that this uses no further evaluations of f. From (39), we see that

$$|\, \mathrm{Int}_d\, f - \mathrm{Int}_{d,\varepsilon}\, f\,| = |\, \mathrm{Int}_d(u - \tilde{u}_\varepsilon)| \le \|u - \tilde{u}_\varepsilon\|_{H^1(I^d)} \le \varepsilon \le \varepsilon \| \mathrm{Int}_d \|_{[H(K_d)]^*}.$$

Since $f \in H_d$ is arbitrary, we see that the algorithm $\mathrm{Int}_{d,\varepsilon}$ produces an ε-approximation to Int_d under the normalized error criterion, using at most $\mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, \Lambda^{\mathrm{std}})$ evaluations of f. Hence

$$\mathrm{card}^{\mathrm{nor}}(\varepsilon, \mathrm{Int}_d, \Lambda^{\mathrm{std}}) \le \mathrm{card}^{\mathrm{abs}}(\varepsilon, S_d, \Lambda^{\mathrm{std}}).$$

Since $(\mathrm{Int}_d, \Lambda^{\mathrm{std}})$ is not weakly tractable under the normalized error criterion, it now follows that $(S_d, \Lambda^{\mathrm{std}})$ is not weakly tractable under the absolute error criterion. □

Theorem 7.1 tells us that sublinear boundedness of the sum of the weights (38) is sufficient for our Neumann problem to be weakly tractable for product weights. Moreover, it also gives us a master kernel K such that sublinear boundedness is necessary for weak tractability. On the other hand, there are some kernels for which this condition is not needed for any level of tractability; for example, if the master kernel K is constant, then $H(K_d)$ is one-dimensional, rendering the Neumann problem (trivially) strongly tractable. It would be useful to characterize those master kernels for which sublinear boundedness of the sum of the weights is necessary and sufficient for weak tractability.

Although we need only assume that the sum of the weights is sublinearly bounded to infer that our problem is weakly tractable, it is reasonable to expect that a stronger condition on the sum of the weights would yield a stronger level of tractability.

We first ask what happens when the sum of the weights is *logarithmically bounded*, which means that

$$a_{\boldsymbol{\gamma}} := \limsup_{d\to\infty} \frac{1}{\ln(d+1)} \sum_{j=1}^{d} \gamma_{d,j} < \infty. \tag{40}$$

Using the notation of (37), we have

Theorem 7.2. *Suppose that the sum of the weights is logarithmically bounded. Let $k \in \{\text{all}, \text{std}\}$. Then the Neumann problem is tractable, with*

$$\begin{aligned} p_{\text{err}}(\Lambda^k) &= \frac{1}{r_k}, \\ p_{\text{dim}}^{\text{abs}}(\Lambda^k) &= 1 + \frac{s_{k,\text{abs}} + 1 + \frac{1}{2}\kappa_0 a_{\boldsymbol{\gamma}}}{r_k} + \delta, \\ p_{\text{dim}}^{\text{nor}}(\Lambda^k) &= 1 + \frac{s_{k,\text{nor}} + \frac{1}{2} + \frac{1}{2}(\kappa_0+\kappa_1) a_{\boldsymbol{\gamma}}}{r_k} + \delta \end{aligned}$$

where $a_{\boldsymbol{\gamma}}$ is as in (40) *and δ is any arbitrary positive number.*

Proof. Since the sum of the weights is logarithmically bounded, it follows that for any $\delta > 0$, there exists a positive integer d_δ such that

$$\frac{1}{\ln(d+1)} \sum_{j=1}^{d} \gamma_{d,j} \le a_{\boldsymbol{\gamma}} + \delta \qquad \forall\, d \ge d_\delta,$$

whence

$$\sigma_d(\theta) \le (d+1)^{\theta(a_{\boldsymbol{\gamma}}+\delta)/2} \qquad \forall\, \theta \ge 0, d \ge d_\delta + 1. \tag{41}$$

Using Lemma 6.1, we now see that

$$C_{d,\text{Lip}} \le \eta_3 d \max\{\sigma_d(\kappa_0), \sigma_{d-1}(\kappa_0)\} \le \eta_2 d (d+1)^{\kappa_0(a_{\boldsymbol{\gamma}}+\delta)/2)} \qquad \forall\, d \ge d_\delta + 1,$$

and so we can take

$$t_{\text{Lip}} = 1 + \tfrac{1}{2}\kappa_0 a_{\boldsymbol{\gamma}} + \delta \qquad \forall\, \delta > 0.$$

For the absolute error criterion, we always have $u_{\text{abs}} = 0$. For the normalized error criterion, Lemma 6.1 and (41) tells us that

$$e(0, \text{App}_{d,2}) = \| \text{App}_{d,2} \|_{\text{Lin}[H(K_d), L_2(I^d)]} \le \sigma_d(\kappa_1) \le (d+1)^{\kappa_1(a_{\boldsymbol{\gamma}}+\delta)/2} \qquad \forall\, d \ge d_\delta. \tag{42}$$

Using Lemma 6.1 along with the fact that $\sigma_d(\theta) \ge 1$ for product weights, we find that

$$e(0, S_d) \ge \frac{\sqrt{2d}\,\sigma_{d-1}(\kappa_2)}{\max\{1, q_0\}} \ge \frac{\sqrt{2d}}{\max\{1, q_0\}}, \tag{43}$$

and so we can take

$$u_{\text{nor}} = \tfrac{1}{2}\kappa_1 a_{\boldsymbol{\gamma}} - \tfrac{1}{2} + \delta \qquad \forall\, \delta > 0.$$

The Theorem follows from Lemma 7.1, along with our values for t_{Lip}, u_{abs}, and u_{nor}. □

Using this theorem, we easily see that if the sum of the weights is logarithmically bounded, then the Neumann problem is always tractable.

Corollary 7.1. *Suppose that* (40) *holds. Then the Neumann problem is tractable, and we may take*

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 3 + (\kappa_0 + \kappa_1) a_{\boldsymbol{\gamma}} + \delta, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 2 + (\kappa_0 + 2\kappa_1) a_{\boldsymbol{\gamma}} + \delta, \end{aligned}$$

and

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 4, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 5 + 2(\kappa_0 + \kappa_1) a_{\boldsymbol{\gamma}} + \delta, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 3 + 2(\kappa_0 + 2\kappa_1) a_{\boldsymbol{\gamma}} + \delta \end{aligned}$$

for any $\delta > 0$.

Proof. From Lemma 4.1, we see that the hypotheses of Theorem 7.2 hold with $r_{\mathrm{all}} = \frac{1}{2}$, $r_{\mathrm{std}} = \frac{1}{4}$, and $s_{\mathrm{all,nor}} = s_{\mathrm{std,nor}} = 0$. Moreover, (42) implies that $s_{\mathrm{all,nor}} = s_{\mathrm{std,nor}} = \frac{1}{2}\kappa_1 a_{\boldsymbol{\gamma}} + \delta$ for any $\delta > 0$. The results now follow from Theorem 7.2. □

We next ask what happens when the sum of the weights is *uniformly bounded*, meaning that

$$a_{\boldsymbol{\gamma}}^* := \sup_{d \in \mathbb{Z}^{++}} \sum_{j=1}^{d} \gamma_{d,j} < \infty. \tag{44}$$

Of course, uniform boundedness implies logarithmic boundedness. Hence Theorem 7.2 immediately tells us that our problem is tractable. However if we use the fact that the sum of the weights is uniformly bounded, we can get a smaller value for the d-exponent.

Theorem 7.3. *Suppose that the sum of the weights is uniformly bounded. Let* $k \in \{\mathrm{all}, \mathrm{std}\}$. *Then the Neumann problem is tractable, with*

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^k) &= \frac{1}{r_k}, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^k) &= 1 + \frac{s_{k,\mathrm{abs}} + 1}{r_k}, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^k) &= 1 + \frac{s_{k,\mathrm{nor}} + \frac{1}{2}}{r_k}. \end{aligned}$$

Proof. Since the sum of the weights is uniformly bounded, we see that

$$\sigma_d(\theta) = \exp\left(\sum_{j=1}^{d} \ln(1 + \theta\gamma_{d,j})\right)^{1/2} \le \exp\left(\sum_{j=1}^{d} \theta\gamma_{d,j}\right)^{1/2} \le e^{a_{\boldsymbol{\gamma}}^* \theta/2} \qquad \forall\, \theta \ge 0, d \in \mathbb{Z}^{++}.$$

Hence

$$C_{d,\mathrm{Lip}} \le \eta_2 d \max\{\sigma_d(\kappa_0), \sigma_{d-1}(\kappa_0)\} \le e^{a_{\boldsymbol{\gamma}}^* \kappa_0/2} \eta_2\, d \qquad \forall\, d \in \mathbb{Z}^{++},$$

and so $t_{\mathrm{Lip}} = 1$. For the absolute error criterion, we always have $u_{\mathrm{abs}} = 0$. For the normalized error criterion, we may use (43) and

$$e(0, \mathrm{App}_{d,2}) = \| \mathrm{App}_{d,2} \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]} \le \sigma_d(\kappa_1) \le e^{a_{\boldsymbol{\gamma}}^* \kappa_1/2} \qquad \forall\, d \in \mathbb{Z}^{++} \tag{45}$$

to see that $u_{\mathrm{nor}} = -\frac{1}{2}$. The results now follow from Lemma 7.1. □

Since uniform boundedness implies logarithmic boundedness, Corollary 7.1 already tells us that the Neumann problem is tractable. However, we can use Theorem 7.3 to get better values for the d-exponents than those provided by Corollary 7.1.

Corollary 7.2. *Suppose that* (44) *holds. Then the Neumann problem is tractable, and we may take*

$$p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) = 2,$$
$$p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) = 3,$$
$$p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) = 2$$

and

$$p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) = 4,$$
$$p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) = 5,$$
$$p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) = 3.$$

Proof. From Lemma 4.1, we see that the hypotheses of Theorem 7.3 hold with $r_{\mathrm{all}} = \frac{1}{2}$, $r_{\mathrm{std}} = \frac{1}{4}$, and $s_{\mathrm{all,nor}} = s_{\mathrm{std,nor}} = 0$. Moreover, (45) implies that $s_{\mathrm{all,nor}} = s_{\mathrm{std,nor}} = 0$. The results now follow from Theorem 7.3. □

Recall that L_2-approximation is strongly tractable for product weights when the sum of the weights is uniformly bounded, see [16]. One might hope that this would also be true for Neumann problem. We shall explore this question further in §8.

7.2 Results for finite-order weights

In this section, we give tractability results for finite-order weights of order ω.

We briefly note a useful estimate for $C_{d,\mathrm{Lip}}$. Suppose that

$$\gamma_{\max} := \sup_{d\in\mathbb{Z}^{++}} \max_{\mathfrak{u}\subseteq[d]} \gamma_{d,\mathfrak{u}} < \infty. \tag{46}$$

Let

$$\eta_4 = 2\min\{1, \kappa_0^{\omega}\}\gamma_{\max},$$

so that

$$\sigma_d(\kappa_1) \le \sigma_d(\kappa_0) \le \sqrt{\eta_4 d^{\omega}}$$

by [19, Lemma 6] and the monotonicity of σ_d. Using (36) and the monotonicity of σ_d, we see that

$$C_{d,\mathrm{Lip}} \le \eta_2\sqrt{\eta_4}\, d^{\omega/2+1}. \tag{47}$$

We will need no further conditions to prove tractability results for finite-order weights. However, we will be able to prove stronger results for the normalized error criterion under the condition $\kappa_2 > 0$ if we make one further assumption, namely, that there exists $c^* > 0$, independent of d, such that

$$\|\,\mathrm{App}_d\,\|_{[H(K_d)]^*} \le c^* \|\,\mathrm{App}_{d-1}\,\|_{[H(K_{d-1})]^*} \tag{48}$$

Remark 7.1. Condition (48) means that $L_2(I^d)$-approximation is not much harder than $L_2(I^{d-1})$-approximation. Should this not be true, we could "reduce" $L_2(I^{d-1})$-approximation problem to $L_2(I^d)$-approximation; this would involve treating a $(d-1)$-variate function in $H(K_{d-1})$ as a d-variate function in $H(K_d)$ that happens to not depend on x_d. So, condition (48) is fairly natural. Note that from Lemma 6.1, we see that (48) is equivalent to the condition

$$\sigma_d(\kappa_1) \le c^*\sigma_{d-1}(\kappa_1). \tag{49}$$

In other words, this condition is a statement about the weights γ. Note that (49) clearly holds if

$$\gamma_{d,\mathfrak{u}} \le c^*\gamma_{d-1,\mathfrak{u}} \qquad \forall\, \mathfrak{u} \in [d-1].$$

Of course, this latter condition is only a sufficient condition for (49) to hold; it is not necessary. □

Theorem 7.4. *Suppose that γ is a family of finite-order weights satisfying* (46) *and having order ω. Let $k \in \{\text{all}, \text{std}\}$. Then*

$$p_{\text{err}}(\Lambda^k) = \frac{1}{r_k},$$

$$p_{\text{dim}}^{\text{abs}}(\Lambda^k) = 1 + \frac{s_{k,\text{abs}} + \frac{1}{2}\omega + 1}{r_k},$$

$$p_{\text{dim}}^{\text{nor}}(\Lambda^k) = \begin{cases} 1 + \dfrac{s_{k,\text{nor}} + \frac{1}{2}(\omega+1)}{r_k} & \textit{if either } \kappa_2 = 0 \textit{ or } (48) \textit{ holds}, \\ \\ 1 + \dfrac{s_{k,\text{nor}} + \frac{1}{2}\omega + 1}{r_k} & \textit{otherwise}. \end{cases}$$

Proof. We once again use the notation of Lemma 7.1. From (47), we see that $t_{\text{Lip}} = \frac{1}{2}\omega + 1$. Since $u_{\text{abs}} = 0$, we immediately see that the result for $p_{\text{dim}}^{\text{abs}}$ holds. It remains to determine u_{nor}.

1. Suppose that $\kappa_2 = 0$. From Lemma 6.1, we see that

$$\frac{\max\{\| \text{App}_{d,2} \|_{\text{Lin}[H(K_d),L_2(I^d)]}, \| \text{App}_{d-1,2} \|_{\text{Lin}[H(K_{d-1}),L_2(I^{d-1})]}\}}{e(0, S_d)} \le \frac{\gamma_{\max}^{1/2} \max\{1, \|W\|_{\text{Lin}[H(K_d)]}\}^{\omega/2}\} \max\{1, q_0\}}{\sqrt{2d}},$$

   and so $u_{\text{nor}} = -\frac{1}{2}$.

2. Now suppose that $\kappa_2 > 0$. From [17, Theorem 2] and Lemma 6.1, we see that

$$\frac{e(0, \text{App}_{d-1,2})}{e(0, S_d)} \le \frac{\max\{1, q_0\}}{\sqrt{2d}} \frac{\sigma_{d-1}(\kappa_1)}{\sigma_{d-1}(\kappa_2)} \le \frac{\max\{1, q_0\}}{\sqrt{2d}} \left(\frac{\omega_1}{\omega_2}\right)^{\omega/2}.$$

   If (48) holds, then

$$\frac{e(0, \text{App}_{d,2})}{e(0, S_d)} \le \frac{c^* \| \text{App}_{d-1,2} \|_{\text{Lin}[H(K_{d-1}),L_2(I^{d-1})]}}{e(0, S_d)} \le \frac{c^* \max\{1, q_0\}}{\sqrt{2d}} \left(\frac{\omega_1}{\omega_2}\right)^{\omega/2},$$

   so that $u_{\text{nor}} = \frac{1}{2}$. However if (48) does *not* hold, then we only have

$$\frac{e(0, \text{App}_{d,2})}{e(0, S_d)} \le \frac{\sigma_d(\kappa_1)}{\sigma_d(\kappa_2)} \le \left(\frac{\omega_1}{\omega_2}\right)^{\omega/2},$$

   and so $u_{\text{nor}} = 0$.

The theorem now follows from Lemma 7.1, along with these values for t_{Lip}, u_{abs}, and u_{nor}. □

Using this theorem, we easily see that if finite-order weights are used, then the Neumann problem is always tractable.

Corollary 7.3. *Suppose that γ is a family of finite-order weights satisfying* (46) *and having order ω. Then the Neumann problem is tractable, and we may take*

$$p_{\text{err}}(\Lambda^{\text{all}}) = 2,$$

$$p_{\text{dim}}^{\text{abs}}(\Lambda^{\text{all}}) = \begin{cases} 3 + \omega & \textit{if } \kappa_2 = 0, \\ 3 + 2\omega & \textit{if } \kappa_2 > 0, \end{cases}$$

$$p_{\text{dim}}^{\text{nor}}(\Lambda^{\text{all}}) = \begin{cases} 2 + \omega & \textit{if either } \kappa_2 = 0 \textit{ or } (48) \textit{ holds}, \\ 3 + \omega & \textit{otherwise} \end{cases}$$

and

$$p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) = 4,$$

$$p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) = \begin{cases} 5 + 2\omega & \text{if } \kappa_2 = 0, \\ 5 + 4\omega & \text{if } \kappa_2 > 0. \end{cases}$$

$$p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) = \begin{cases} 3 + 2\omega & \text{if either } \kappa_2 = 0 \text{ or (48) holds}, \\ 5 + 2\omega & \text{otherwise}. \end{cases}$$

Proof. From Lemma 4.1, we see that the hypotheses of Theorem 7.4 hold with $r_{\mathrm{all}} = \frac{1}{2}$, $r_{\mathrm{std}} = \frac{1}{4}$, and $s_{\mathrm{all,nor}} = s_{\mathrm{std,nor}} = 0$. We need only determine $s_{\mathrm{all,abs}}$ and $s_{\mathrm{std,abs}}$.

1. If $\kappa_2 = 0$, then Lemma 6.1 tells us that

$$e(0, \mathrm{App}_{d,2}) = \| \mathrm{App}_{d,2} \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]} \le \sigma_d(\kappa_1) \le \sqrt{\eta_4 d^{\omega}},$$

   and so $s_{\mathrm{all,abs}} = s_{\mathrm{std,abs}} = \frac{1}{2}\omega$.

2. If $\kappa_2 > 0$, then Lemma 6.1 tells us that

$$e(0, \mathrm{App}_{d,2}) = \| \mathrm{App}_{d,2} \|_{\mathrm{Lin}[H(K_d), L_2(I^d)]} \le \gamma_{\max}^{1/2} \max\{1, \| W \|_{\mathrm{Lin}[H(K_d)]}\}^{\omega/2}\},$$

   and so $s_{\mathrm{all,abs}} = s_{\mathrm{std,abs}} = 0$.

The results now follow from Theorem 7.3. □

In [17], the L_2-approximation was found to be strongly tractable for finite-order weights when $\kappa_2 > 0$, see [17]. As we stated at the end of §7.1, one might hope that this would also be true for the Neumann problem. We shall explore this question further in the next section.

8 Can the nonhomogeneous Neumann problem be strongly tractable?

So far, our best results have established tractability results for the nonhomogeneous Neumann problem, without any strong tractability results. Is this a weakness of our proofs and techniques, or does the Neumann problem fail to be strongly tractable?

We show that the fault lies not in our proofs, but in (the formulation of) our problem. First, we show that the problem is never strongly tractable under the absolute error criterion, whether continuous linear information or standard information is used. Then, we show that if (48) holds and if there exists a positive constant c^{**}, independent of d, such that

$$\| \mathrm{App}_{d-1} \|_{[H(K_d)]^*} \le c^{**} \| \mathrm{Int}_{d-1} \|_{[H(K_{d-1})]^*}, \tag{50}$$

then our problem is not strongly tractable under the normalized error criterion.

Theorem 8.1. *Whether we are using continuous linear information or standard information, the following hold:*

1. *The Neumann problem is not strongly tractable for the absolute error criterion.*

2. *If* (48) *and* (50) *hold, then the Neumann problem is not strongly tractable under the normalized error criterion.*

Proof. In what follows, we let Λ denote either Λ^{all} or Λ^{std}, as appropriate.

First, we claim that

$$e(n, S_d, \Lambda) \ge \frac{(2d - n) \| \mathrm{Int}_{d-1} \|_{[H(K_{d-1})]^*}}{\max\{1, q_0\}} \qquad \text{if } n < 2d. \tag{51}$$

Indeed, let N be information of cardinality $n < 2d$, so that our sole knowledge about the problem instance $[f, g, q] \in H_d \times \widetilde{H}_{d-1} \times (Q_d \cap H_{d,\rho})$ when approximating $S_d(f, g, q)$ is given by $N(f, g, q)$. There must be at least $2d - n$ faces $I^d_{j_1,\theta_1}, \ldots, I^d_{j_{2d-n},\theta_{2d-n}}$ of I^d at which N does not sample boundary data. Let $z \in H_{d-1}$. Define $g\colon \partial I^d \to \mathbb{R}$ as

$$g(\mathbf{x}) = \begin{cases} (-1)^\theta z(\hat{\mathbf{x}}_{[j_i]}) & \text{if } \mathbf{x} \in I^d_{j_i,\theta_i} \text{ for some } i \in \{1, \ldots, 2d-n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in H_d$ and $N([0, g, 1]) = 0$. Let $u = S_d(0, g; q_0)$. Then

$$\max\{1, q_0\}\|u\|_{H^1(I^d)} \geq B_d(u, 1; q_0) = \langle g, 1\rangle_{L_2(\partial I^d)} = (2d - n)\int_{I^{d-1}} z(\mathbf{y})\, d\mathbf{y}. \tag{52}$$

Let $e(N, S_d, \Lambda)$ be the minimal error among all algorithms using the information N. From [15, §4.5] and (52), we see that

$$e(N, S_d, \Lambda) \geq \|u\|_{H^1(I^d)} \geq \frac{(2d-n)}{\max\{1, q_0\}}\int_{I^{d-1}} z(\mathbf{y})\, d\mathbf{y}.$$

Since this is true for any $z \in H_{d-1}$, we find that

$$e(N, S_d, \Lambda) \geq (2d - n)\frac{\|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*}}{\max\{1, q_0\}}.$$

Finally, since N is arbitrary information of cardinality at most $n < 2d$, inequality (51) holds, as claimed.

We now consider the absolute error criterion. Recall the definition of $\gamma_{\min,\emptyset}$ from (33). For $d \in \mathbb{Z}^{++}$, let

$$\varepsilon_0 = \min\left\{\frac{\gamma_{\min,\emptyset}}{\max\{1, q_0\}}, \tfrac{1}{2}\right\},$$

noting that $\varepsilon_0 \in (0, 1)$. By Lemma 6.1 and the definition of σ_d, we have

$$\|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*} = \sigma_{d-1}(\kappa_2) \geq \gamma_{\min,\emptyset}.$$

Letting $n = 2d - 1$ in (51), we see that

$$e(2d-1, S_d, \Lambda) \geq \frac{\gamma_{\min,\emptyset}}{\max\{1, q_0\}} \geq \varepsilon_0.$$

and so

$$\operatorname{card}^{\text{abs}}(\varepsilon_0, S_d, \Lambda) \geq 2d.$$

As a result, it follows that $\operatorname{card}^{\text{abs}}(\varepsilon, S_d, \Lambda)$ cannot be bounded from above by a function of ε for all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{Z}^{++}$.

Finally, we consider the normalized error criterion under the condition that (48) and (50) hold. Without loss of generality, we assume that

$$c^{**}(c^* + \sqrt{5}) > 2.$$

Let

$$\varepsilon_1 = \frac{\min\{1, q_0\}}{\max\{1, q_0\}}\frac{1}{c^{**}(c^* + \sqrt{5})},$$

noting that $\varepsilon_1 \in (0, 1)$. From Theorem 3.3 and (51) with $n = 2d - \lceil\sqrt{2d}\rceil$, we find that

$$\begin{aligned}\frac{e(2d - \lceil\sqrt{2d}\rceil, S_d, \Lambda)}{e(0, S_d)} &\geq \frac{\min\{1, q_0\}}{\max\{1, q_0\}}\frac{\sqrt{2d}\,\|\operatorname{Int}_{d-1}\|_{[H(K_{d-1})]^*}}{\|\operatorname{App}_{d,2}\|_{\operatorname{Lin}[H(K_d), L_2(I^d)]} + \sqrt{2d+1}\,\|\operatorname{App}_{d-1,2}\|_{\operatorname{Lin}[H(K_{d-1}), L_2(I^{d-1})]}} \\ &\geq \frac{\min\{1, q_0\}}{\max\{1, q_0\}}\frac{\sqrt{2d}}{c^{**}(c^* + \sqrt{2d+1})} \geq \varepsilon_1.\end{aligned}$$

Hence

$$\mathrm{card}^{\mathrm{nor}}(\varepsilon_1, S_d, \Lambda) \geq 2d - \lceil \sqrt{d} \rceil.$$

As a result, it follows that $\mathrm{card}^{\mathrm{nor}}(\varepsilon, S_d, \Lambda)$ cannot be bounded from above by a function of ε for all $\varepsilon \in (0, 1)$ and all $d \in \mathbb{Z}^{++}$. □

Remark 8.1. We have already discussed the condition (48) in Remark 7.1. What can we say about (50)? This condition says that integration is not much easier than approximation. From Lemma 6.1, we know that (50) holds if

$$\sigma_{d-1}(\kappa_1) \leq c^{**} \sigma_{d-1}(\kappa_2). \tag{53}$$

So when does (53) hold?

1. Suppose we are using product weights. Then

$$\frac{\sigma_{d-1}(\kappa_1)}{\sigma_{d-1}(\kappa_2)} = \left[\prod_{j=1}^{d-1} \frac{1+\kappa_1\gamma_{d-1,j}}{1+\kappa_2\gamma_{d-1,j}}\right]^{1/2} \leq \left[\prod_{j=1}^{d-1} \big(1+(\kappa_1-\kappa_2)\gamma_{d-1,j}\big)\right]^{1/2} = \sigma_{d-1}(\kappa_1-\kappa_2).$$

   Suppose that the bounded-sum condition (44) holds. Then

$$\frac{\sigma_{d-1}(\kappa_1)}{\sigma_{d-1}(\kappa_2)} \leq e^{a_\gamma^*(\kappa_1-\kappa_2)/2},$$

   and so (50) holds with $c^{**} = e^{a_\gamma^*(\kappa_1-\kappa_2)/2}$. However if the bounded-sum condition (44) does not hold, then (53) does not hold.

2. Suppose we are using finite-order weights of order ω. If $\kappa_2 > 0$, then [17, Theorem 2] tells us that

$$\frac{\sigma_{d-1}(\kappa_1)}{\sigma_{d-1}(\kappa_2)} \leq \left(\frac{\kappa_1}{\kappa_2}\right)^{\omega/2},$$

   and so (50) holds with $c^{**} = (\kappa_1/\kappa_2)^{\omega/2}$. However if $\kappa_2 = 0$, then condition (53) does not hold. □

So the Neumann problem is never strongly tractable for the absolute error criterion. Moreover, our problem is not strongly tractable for the normalized error criterion, provided that the conditions (48) and (50) hold. We conjecture that our problem is never strongly tractable for the normalized error criterion.

9 Some illustrations

Up to this point, we have given results that hold for any reproducing kernel K satisfying our conditions (26)–(28). In this section, we give tractability exponents for two specific kernels: the Korobov kernel $K_{\mathrm{Kor},r}$ and the min kernel $K_{\min}$. Our results are based on those found in [9]. We will only discuss product weights, since these were the only weights that [9] analyzed.

From our general results in §7, we know that weak tractability depends on whether or not the sum of the weights is sublinearly bounded. Since this is a cut-and-dried "yes-or-no" condition, there is nothing further to add when discussing specific reproducing kernels. From the results in §8, we know that our problem is not strongly tractable. So it only remains to determine the ε- and d-exponents of tractability.

As in [9], we define the sum exponents

$$s_\gamma := \inf\left\{ s > 0 : \sup_{d \in \mathbb{Z}^{++}} \sum_{j=1}^{d} \gamma_{d,j}^s < \infty \right\}$$

and

$$t_\gamma := \inf\left\{ t > 0 : \sup_{d\in\mathbb{Z}^{++}} \sum_{j=1}^{d} \frac{\gamma_{d,j}^t}{\ln(d+1)} < \infty \right\},$$

with the convention that $\inf \emptyset = 0$. We also define

$$R_\tau := \limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_{d,j}^\tau}{\ln(d+1)} \qquad \forall\, \tau \in (0,1].$$

Note that if the sum of the weights is logarithmically bounded, then $t_\gamma \le 1$ and $R_\tau < \infty$ for $\tau \in [0, t_\gamma]$. Moreover if the sum of the weights is bounded, then $s_\gamma \le 1$. We shall let ζ denote the usual Riemann zeta function.

We first consider the Korobov kernel.

Theorem 9.1. *Suppose that $K = K_{\mathrm{Kor},r}$ for some $r \in \mathbb{Z}^{++}$.*

1. *Suppose that the sum of the weights is logarithmically bounded, with a_γ as in* (40)*. For $t_\gamma = 1$, take $\tau = 1$, and for $t_\gamma < 1$, let τ be any element of* $(\max\{1/(2r), t_\gamma\}]$*. For continuous linear information, we may take*

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2\tau, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 1 + \big(2 + (\kappa_0+\kappa_1)a_\gamma\big)\tau + 2\zeta(2r\tau) + \delta \qquad \forall\, \delta > 0, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 1 + \big(1 + (\kappa_0+\kappa_1)a_\gamma\big)\tau + 2\zeta(2r\tau) + \delta \qquad \forall\, \delta > 0, \end{aligned}$$

and for standard information, we may take

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 2\tau(1+\tau), \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 1 + \big[\big(2 + (\kappa_0+\kappa_1)a_\gamma\big)\tau + 2\zeta(2r\tau)\big](1+\tau) + \delta \qquad \forall\, \delta > 0, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 1 + \big[\big(1 + (\kappa_0+\kappa_1)a_\gamma\big)\tau + 2\zeta(2r\tau)\big](1+\tau) + \delta \qquad \forall\, \delta > 0. \end{aligned}$$

2. *Suppose that the sum of the weights is uniformly bounded. For $s_\gamma = 1$, take $\tau = 1$, and for $s_\gamma < 1$, let τ be any element of* $(\max\{1/(2r), s_\gamma\}]$*. For continuous linear information, we may take*

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2\tau, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 1 + 2\tau, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 1 + \tau, \end{aligned}$$

and for standard information, we may take

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 2\tau(1+\tau), \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 1 + 2\tau(1+\tau) + \delta \qquad \forall\, \delta > 0, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 1 + \tau(1+\tau) + \delta \qquad \forall\, \delta > 0. \end{aligned}$$

Proof. We first suppose that the sum of the weights is logarithmically bounded. Since (42) holds, we see that we may take

$$s_{k,\mathrm{abs}} = s_{k,\mathrm{nor}} + \tfrac{1}{2}\kappa_1 a_\gamma + \delta \qquad \forall\, \delta > 0,\ k \in \{\mathrm{all}, \mathrm{std}\}.$$

From [9, Theorem 5], we have

$$r_{\mathrm{all}} = \frac{1}{2\tau} \qquad \text{and} \qquad r_{\mathrm{std}} = \frac{1}{2\tau(1+\tau)} + \delta \qquad \forall\, \delta > 0$$

and that

$$s_{\mathrm{all,nor}} = s_{\mathrm{std,nor}} = \frac{\zeta(2r\tau)R_\tau}{\tau} + \delta \qquad \forall\, \delta > 0. \tag{54}$$

The result now follows from Theorem 7.2.

Now suppose that the sum of the weights is uniformly bounded. Since (45) holds, we may again use [9, Theorem 5] to see that

$$s_{k,\mathrm{abs}} = s_{k,\mathrm{nor}} = 0 \qquad \text{for } k \in \{\mathrm{all}, \mathrm{std}\}$$

and that

$$r_{\mathrm{all}} = \frac{1}{2\tau} \qquad \text{and} \qquad r_{\mathrm{std}} = \frac{1}{2\tau(1+\tau)} + \delta \qquad \forall\, \delta > 0.$$

The result once again follows from Theorem 7.2. □

It is natural to compare the results of Theorem 9.1 with the results of §7. In particular, what happens if we let $\tau = 1$ in Theorem 9.1?

1. When the sum of the weights is logarithmically bounded, we find

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 3 + (\kappa_0 + \kappa_1)a_\gamma + 2R_1\zeta(2r) + \delta \qquad \forall\, \delta > 0, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 2 + (\kappa_0 + \kappa_1)a_\gamma + 2R_1\zeta(2r) + \delta \qquad \forall\, \delta > 0, \end{aligned}$$

   and

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 4, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 5 + 2(\kappa_0 + \kappa_1)a_\gamma + 4R_1\zeta(2r) + \delta \qquad \forall\, \delta > 0, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 3 + 2(\kappa_0 + \kappa_1)a_\gamma + 4R_1\zeta(2r) + \delta \qquad \forall\, \delta > 0, \end{aligned}$$

   which is somewhat worse than the results reported in Corollary 7.1, since these latter results contain a term involving $R_1\zeta(2r)$.

2. When the sum of the weights is uniformly bounded, we find

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 3, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 2, \end{aligned}$$

   and

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 4, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 5 + \delta \qquad \forall\, \delta > 0, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 3 + \delta \qquad \forall\, \delta > 0, \end{aligned}$$

   which is comparable to the results contained in Corollary 7.2.

Next, we consider the min kernel. Let

$$b_\tau = \frac{\zeta(2\tau)}{\pi^{2\tau}} + \frac{1}{2^\tau} \qquad \forall\, \tau > 0.$$

Theorem 9.2. *Suppose that $K = K_{\min}$.*

1. *Suppose that the sum of the weights is logarithmically bounded, with a_γ as in* (40)*. For $t_\gamma = 1$, take $\tau = 1$, and for $t_\gamma < 1$, let τ be any element of* $(\max\{\frac{1}{2}, t_\gamma\}]$*. Then for continuous linear information, we may take*

$$\begin{aligned} p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2\tau, \\ p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 1 + \big(2 + (\kappa_0 + \kappa_1)a_\gamma\big)\tau + 2b_\tau R_\tau + \delta \qquad \forall\, \delta > 0, \\ p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 1 + \big(1 + (\kappa_0 + \kappa_1)a_\gamma\big)\tau + 2b_\tau R_\tau + \delta \qquad \forall\, \delta > 0, \end{aligned}$$

and for standard information, we may take

$$\begin{aligned}
p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 2\tau(1+\tau), \\
p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 1 + \big[\big(2 + (\kappa_0+\kappa_1)a_{\boldsymbol{\gamma}}\big)\tau + 2b_\tau R_\tau\big](1+\tau) + \delta \qquad \forall\,\delta > 0, \\
p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 1 + \big[\big(1 + (\kappa_0+\kappa_1)a_{\boldsymbol{\gamma}}\big)\tau + 2b_\tau R_\tau\big](1+\tau) + \delta \qquad \forall\,\delta > 0.
\end{aligned}$$

2. *Suppose that the sum of the weights is uniformly bounded. For $s_{\boldsymbol{\gamma}} = 1$, take $\tau = 1$, and for $s_{\boldsymbol{\gamma}} < 1$, let τ be any element of* $(\max\{1/(2r), s_{\boldsymbol{\gamma}}\}]$. *Then for continuous linear information, we may take*

$$\begin{aligned}
p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2\tau, \\
p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 1 + 2\tau, \\
p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 1 + \tau,
\end{aligned}$$

and for standard information, we may take

$$\begin{aligned}
p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 2\tau(1+\tau), \\
p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 1 + 2\tau(1+\tau) + \delta \qquad \forall\,\delta > 0, \\
p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 1 + \tau(1+\tau) + \delta \qquad \forall\,\delta > 0.
\end{aligned}$$

Proof. The proof is almost the same as that of Theorem 9.1, except that we now use [9, Theorem 7], rather than [9, Theorem 5]. The only difference between these two results of [9] is that whereas we had (54) for the Korobov kernel, we have

$$s_{\mathrm{all,nor}} = s_{\mathrm{std,nor}} = \frac{b_\tau R_\tau}{\tau} + \delta \qquad \forall\,\delta > 0$$

for the min kernel. □

Once again, it is natural to compare the results of Theorem 9.2 with the results of §7. In particular, what happens if we let $\tau = 1$ in Theorem 9.2?

1. When the sum of the weights is logarithmically bounded, we find

$$\begin{aligned}
p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2, \\
p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 3 + (\kappa_0+\kappa_1)a_{\boldsymbol{\gamma}} + 2b_1R_1 + \delta \qquad \forall\,\delta > 0, \\
p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 2 + (\kappa_0+\kappa_1)a_{\boldsymbol{\gamma}} + 2b_1R_1 + \delta \qquad \forall\,\delta > 0,
\end{aligned}$$

   and

$$\begin{aligned}
p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 4, \\
p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 5 + 2(\kappa_0+\kappa_1)a_{\boldsymbol{\gamma}} + 4b_1R_1 + \delta \qquad \forall\,\delta > 0, \\
p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 4 + 2(\kappa_0+\kappa_1)a_{\boldsymbol{\gamma}} + 4b_1R_1 + \delta \qquad \forall\,\delta > 0,
\end{aligned}$$

   which is somewhat worse than the results reported in Corollary 7.1, since these latter results contain a term involving b_1R_1.

2. When the sum of the weights is uniformly bounded, we find

$$\begin{aligned}
p_{\mathrm{err}}(\Lambda^{\mathrm{all}}) &= 2, \\
p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{all}}) &= 3, \\
p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{all}}) &= 2,
\end{aligned}$$

and

$$
\begin{aligned}
p_{\mathrm{err}}(\Lambda^{\mathrm{std}}) &= 4, \\
p_{\mathrm{dim}}^{\mathrm{abs}}(\Lambda^{\mathrm{std}}) &= 5+\delta \qquad \forall\, \delta > 0, \\
p_{\mathrm{dim}}^{\mathrm{nor}}(\Lambda^{\mathrm{std}}) &= 3+\delta \qquad \forall\, \delta > 0,
\end{aligned}
$$

which is comparable to the results contained in Corollary 7.2.

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