Rigid Formations with Leader-Follower Architecture

Tolga Eren, Walter Whiteley, and Peter Belhumeur, *Member, IEEE*

Abstract

This paper is concerned with information structures used in rigid formations of autonomous agents that have leader-follower architecture. The focus of the paper is on sensor/network topologies to secure control of rigidity. This papers extends the previous rigidity based approaches for formations with symmetric neighbor relations to include formations with leader-follower architecture. We provide necessary and sufficient conditions for rigidity of directed formations, with or without cycles. We present the directed Henneberg constructions as a sequential process for all guide rigid digraphs. We refine those results for acyclic formations, where guide rigid formations had a simple construction. The analysis in this paper confirms that acyclicity is not a necessary condition for stable rigidity. The cycles are not the real problem, but rather the lack of guide freedom is the reason behind why cycles have been seen as a problematic topology. Topologies that have cycles within a larger architecture can be stably rigid, and we conjecture that all guide rigid formations are stably rigid for internal control. We analyze how the external control of guide agents can be integrated into stable rigidity of a larger formation. The analysis in the paper also confirms the inconsistencies that result from noisy measurements in redundantly rigid formations. An algorithm given in the paper establishes a sequential way of determining the directions of links from a given undirected rigid formation so that the necessary and sufficient conditions are fulfilled.

Index Terms

Multi-agent systems, robot formations, rigid formations, graph theory, rigidity theory.

Regular Paper

December 1, 2005

T. Eren is the corresponding author. He is with the Department of Electrical Engineering, Kirikkale University, Kirikkale, Turkey.
Address: Arifiye mh. Alaeddin cd. No:12 Akturk Apt. Daire:12, Kat:4, 26010 Eskisehir, Turkey. E-mail: tolga.eren@aya.yale.edu.

W. Whiteley is with the Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada.
Address: 4700 Keele Street, Toronto, Ontario Canada M3J 1P3. E-mail: whiteley@mathstat.yorku.ca.

P. Belhumeur is with the Computer Science Department, Columbia University, New York NY 10027 USA.
Address: Computer Science Department, Columbia University, 500 West 120th Street, New York, NY 10027 USA. E-mail: belhumeur@cs.columbia.edu.

T. Eren and P. Belhumeur are supported by the National Science Foundation under grants NSF ITR IIS-03-25864, and NSF IIS-03-08185.
W. Whiteley is supported by the Natural Science and Engineering Research Council (Canada) and the National Institutes of Health (USA).

Rigid Formations with Leader-Follower Architecture

I. Introduction

Multi-agent systems have lately received considerable attention due to recent advances in computation and communication technologies (see for example [1]–[7]). In this paper, agents will simply be thought of as autonomous agents including robots, unmanned aerial vehicles, microsatellites, ground vehicles, autonomous underwater vehicles, and sensor nodes. A *formation* is a group of agents moving in real 2- or 3-space, with some specified links whose distances are maintained. A formation is called *rigid* if the distance between each pair of agents does not change over time under ideal conditions. A formation is called *minimally rigid* if it loses its rigidity when any one of its links is removed from the formation. In other words, a minimally rigid formation has the minimum number of links to maintain rigidity. If a formation is rigid but not minimally rigid, then it is called a *redundantly rigid* formation.

Since minimally rigid formations have the least number of sensing/communication links for maintaining rigidity, they are more energy-efficient compared to redundantly rigid formations. Moreover, when measurements are noisy, redundant rigidity causes problems in formation realizability, and this issue will be explored in the paper. Sensing/communication links are used for maintaining fixed distances between agents. The interconnection structure of sensing/communication links is called *sensor/network topology*. In practice, actual agent groups cannot be expected to move exactly as a rigid formation because of sensing errors, actuation errors, actuation delays, vehicle modelling errors, etc. The ideal benchmark formation against which the performance of an actual agent formation is to be measured is called a *reference formation*.

In reality, agents are entities with physical dimensions. For modeling purposes in this paper, agents are represented by points called *point agents*. Distances between all agent pairs can be held fixed by directly measuring distances between only some agents and keeping them at desired values. A distance constraint or link, is a requirement that a distance between two agents, depicted with d, be maintained through a sensing/communication link and some control strategy. Distance constraints are sometimes referred to as range or separation constraints. With enough distance constraints, the whole formation will be rigid, even without there being a distance constraint between every pair of agents.

Two agents connected by a sensing/communication link are called *neighbors*. There are two types of neighbor relations in rigid formations. In the first type, the neighbor relation is symmetric, i.e., if agent i senses/communicates with agent j and performs action upon the information it receives, so does agent j with agent i. A link with a symmetric neighbor relation is represented graphically by a straight line. In the second type, the neighbor relation is asymmetric, i.e., if agent i senses/communicates with agent j and performs actions upon the information it receives, then agent j does not make use of any information received from agent i although it may sense/communicate with agent i. For example, rigid formations with a leader-follower architecture have the asymmetric neighbor relation. A link with an asymmetric neighbor relation between a leader and a follower is represented by a directed edge, or arrow, pointing from the follower to the leader, i.e., head is the leader and tail is the follower. The terms *undirected formation* and *directed formation* are used throughout the paper to describe formations with symmetric neighbor relations and formations with leader-follower architecture, respectively [3].

The work in [2], [8]–[10] suggested an approach based on rigidity for maintaining formations of autonomous agents with sensor/network topologies that use distance information between agents, where the neighbor relation is symmetric. Rigidity of undirected formations with distance information is well understood in 2-space, and there are partial results in 3-space [10]. Other researchers focused on using both distance and bearing information to maintain formations that have leader-follower architecture [6].

This paper is concerned with directed rigid formations with distance constraints. We restrict our attention to minimally rigid formations in 2-space. We wish to consider a broader range of interconnection topologies, including both cyclic and acyclic, and understand how the interconnection topology and the directions affect the rigidity of a formation as it performs a coordinated motion. Our ultimate goal is the development of strategies to create minimally rigid directed formations, which are scalable for any number of agents. There are a number of issues that must be addressed in order to maintain a rigid directed formation. We identify four key layers as follows:

1) rigidity of the undirected reference formation;
2) guide rigidity of the directed reference formation;
3) stable rigidity, or internal control of the formation;
4) external motion control of the formation or motion planning of agents that have non-zero internal degrees of freedom.

As it will become clear throughout the paper, if underlying undirected formation is non-rigid, then directed formation cannot be rigid, or stable. Thus, undirected rigidity is a necessary condition for directed rigidity. On the other hand, when we associate a direction to each link in a rigid undirected formation, directed rigidity is not necessarily guaranteed. That is, undirected rigidity is not a sufficient condition for directed rigidity.

One can think of reference formation as if all agents act like ideal agents that can perform actions instantly in order to satisfy distance constraints. Joints in a bar-joint framework, which will be explained in the next section, can be considered as an example of such ideal agents and bars are ideal distance constraints. Internal control and stability problems arise as the third layer, because agents cannot act like ideal agents in practice due to actuation errors, delays, errors in measurements, etc. Stability properties of directed formations with acyclic interconnections were studied in [4], and those with linear cyclic interconnections were studied in [11]. Stability properties of acyclic directed rigid formations were also studied in [5]. A directed formation that is both rigid and stable is called a *stably rigid* formation. Reference formations of stably rigid acyclic directed formations with distance information in 2-space are studied in [12] and is further refined here.

The fourth layer is moving the stably rigid formation as a rigid object along an assigned path. This requires sensible external control of agents that have nonzero internal degrees of freedom. If an agent does not maintain any distance constraints to any other agent, then this agent has two degrees of freedom in 2-space, and is called a *global leader*. If an agent maintains one distance constraint to one other agent, then this agent has one degree of freedom, and is a *free follower*. A free follower is called a *first follower* if it is a neighbor of the global leader. The collection of global leaders and free followers make up the set of *guide agents* of a formation. If an agent has to maintain distances to two other agents, then this agent has zero degrees of freedom and is called an ordinary agent. The choice of guide agents determines the distribution of the external degrees of freedom among agents.

It is easy to see that we need rigidity of reference formation before attempting to solve the problem of stable rigidity and the problem of motion planning for guide agents. Therefore, the analysis of a directed rigid formation can be structured sequentially into the four levels: (i) undirected rigidity; (ii) directed rigidity; (iii) stable rigidity; (iv) motion planning of guide agents. We will touch all of these four issues throughout the paper, but the focus of the paper is mainly on the rigidity of directed reference formation.

We note that redundantly rigid formations (with more links than the minimum for rigidity) lead to overdetermined systems of constraints. Inconsistencies in overdetermined systems caused by redundant rigidity and noisy measurements are called *redundancy-based inconsistencies*. These inconsistencies give rise to problems in realizability. Although 2-cycles cause redundancy-based inconsistencies, we will see that cycles of length 3 or more can be internally controlled in minimally rigid formations, where they do not cause redundancy-based inconsistencies.

In particular, it has been asserted that a formation that has the topology of a 3-cycle is not stably rigid [5]. This brought about the generalization that stable rigidity requires acyclicity in rigid formations. However, we will show that such cycles are not a real problem for stability, but are an issue for the next layer of external control. As such they can be included within a larger architecture.

The contributions of this paper are:

1) to extend the previous rigidity based approaches for formations with symmetric neighbor relations to include formations with leader-follower architecture;
2) to give necessary and sufficient conditions for rigidity of directed formations, with or without cycles;
3) to develop techniques for directly creating sensor/network topologies of directed rigid formations;
4) to present an algorithm to determine the directions of links to create a stably rigid formation from a rigid undirected formation;
5) to analyze how noisy measurements cause inconsistencies in redundantly rigid formations;
6) to analyze the behavior of cycles in a leader-follower formation, to distinguish: when they indicate internal control problems, due to redundancy; when they fit stable rigidity in minimally rigid formations; and how they might impact external control through the guide agents;
7) to analyze how the external control of guide agents can be integrated into stable rigidity of a larger formation.

The paper is organized as follows. In §II, we start with definitions of rigidity. We review point formations in §II-A, and rigid formations with symmetric neighbor relations in §II-B. We investigate rigid formations that have leader-follower architecture in §III. Acyclic formations are explored in §IV. Cycles in rigid formations are studied in §V. Finally, concluding remarks are given in §VII.

II. Rigidity and Point Formations

One way of visualizing rigidity with symmetric neighbor relation is to imagine a collection of rigid bars connected to one another by idealized ball joints, which is called a bar-joint framework [13]. By an idealized ball joint we mean a connection between a collection of bars which only imposes the restriction that the bars share common endpoints, and no angle constraints. Now, can the joints be moved in a continuous manner without changing the lengths of any of the bars, where translations and rotations do not count? If so, the framework is flexible; if not, it is rigid. (Precise definitions will appear in the next section.) In a bar-joint framework, the length of a bar imposes a distance constraint for both end-joints. This is modeled in a formation where two agents connected by a sensing/communication link are mutually affected by the information conveyed by this link. For example, if two agents connected by a sensing/communication link are set to maintain a ten meter distance between them, then both agents perform action to maintain this distance. In the graph theoretic setting, the edge corresponding to this link is denoted by an undirected edge.

A. Point Formations

A point formation $\mathbb{F}(p) \triangleq (p, \mathcal{E})$ provides a way of representing a formation of n agents. The configuration $p \triangleq \{p_1, p_2, \ldots, p_n\}$ where the points p_i represent the positions of agents in $\mathbb{R}^2$ and i is an integer in $\{1, 2, \ldots, n\}$ denoting an agent. $\mathcal{E}$ is the set of "maintenance links," labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$. The *maintenance links* in $\mathcal{E}$ correspond to constraints between specific agents, such as distances, which are to be maintained over time by using sensing/communication links between certain pairs of agents. Each point formation $\mathbb{F}(p)$ uniquely determines a graph

$\mathbb{G}_{\mathbb{F}(p)} \triangleq (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$, which is the set of labels of agents, and edge set $\mathcal{E}$. A formation with distance constraints can be represented by $(\mathcal{V}, \mathcal{E}, f)$ where $f : \mathcal{E} \longmapsto \mathbb{R}^+$ measures the length of links. Each maintenance link $(i, j) \in \mathcal{E}$ is used to maintain the distance $f((i, j))$ between certain pairs of agents fixed.

A *trajectory* of a formation is a continuously parameterized one-parameter family of curves $(q_1(t), q_2(t), ..., q_n(t))$ in $\mathbb{R}^{2n}$ which contain p and on which for each t, $\mathbb{F}(q(t))$ is a formation with the same measured values under f. We will say that two point formations $\mathbb{F}(p)$ and $\mathbb{F}(r)$, with $p, r \in q(t)$, are *congruent* if p and r are congruent. That is p is *congruent* to r in the sense that there is a distance-preserving map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(r_i) = p_i, i \in \{1, 2, \ldots, n\}$. A *rigid motion* is a trajectory along which point formations contained in this trajectory are all congruent to the original configuration. If rigid motions are the only possible trajectories then the formation is called *rigid*; otherwise it is called *flexible* [8], [10].

B. Rigidity in Point Formations with Symmetric Neighbor Relations

Whether the formation with symmetric neighbor relations is rigid or not can be studied by the solutions to a system of quadratic equations. In particular, the motion of $\mathbb{F}(p) = (\{p_1, p_2, \ldots, p_n\}, \mathcal{E})$ with m maintenance links, along the trajectory $q([0, \infty)) \triangleq \{\{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ preserves the Euclidean distances $d_{ij} \triangleq ||p_i - p_j||$ between pairs of points (p_i, p_j) for which (i, j) is a link. Along such a trajectory

$$(q_i - q_j) \cdot (q_i - q_j) = d_{ij}^2, \qquad (i, j) \in \mathcal{E}, \quad t \geq 0 \qquad (1)$$

We note that the existence of a trajectory is equivalent to the existence of a piecewise analytic path, with all derivatives at the initial point [14]. Assuming a smooth (piecewise analytic) trajectory, we can differentiate to get

$$(q_i - q_j) \cdot (\dot{q}_i - \dot{q}_j) = 0, \qquad (i, j) \in \mathcal{E}, \quad t \geq 0 \qquad (2)$$

Here, $\dot{q}_i$ is the velocity of point i. The m linear equations are collected into a single matrix equation

$$R(\mathbb{F}; q)\dot{q}^T = 0 \qquad (3)$$

where $\dot{q} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$ and $R(\mathbb{F}; q)$ is a specially structured $m \times 2n$ matrix called the *rigidity matrix* [13], [15], [16]. The following theorem holds [15], [16]:

Theorem 2.1: An n-point formation $\mathbb{F}(p)$ with at least 2 points in 2-space is first-order rigid if and only if

$$\operatorname{rank} R(\mathbb{F}; p) = 2n - 3.$$

Note that $\operatorname{rank} R(\mathbb{F}; p) = 2n - 3$, also called first-order rigidity, implies rigidity for the formation. First-order rigidity is a robust property: a small change in the position of points, or of the length of maintenance links within the neighborhood preserves this rigidity. When the rank of $R(\mathbb{F}; p)$ is the maximum, the behavior will be *generic* for $\mathcal{E}$ (the matrix will have the same rank for almost all positions $p \in \mathbb{R}^{2|V|}$). When the rank is less than the maximum, a pattern $\mathbb{G}_{\mathbb{F}(p)}$ which is rigid for most formations may become flexible and a pattern which is flexible for most formations may become rigid. Both of these non-generic situations will turn out to be unstable for control purposes.

If we have a formation that is not at the generic rank, then the rows of the rigidity matrix will be dependent and there will also be first-order flexes. Therefore some small errors or noise in the lengths will correspond to a pair of nearby realizations that have all the same link lengths, and the formation can easily vibrate between them under internal control. The mathematical process of finding this pair is known as 'averaging' and 'deaveraging' [13]. Other effects of noise in formations that have dependent rigidity matrices will be explored in §V.

1) Generic Rigidity: We define a type of rigidity, called "generic rigidity," that is most useful for our purposes. A set $\mathcal{A} = (\alpha_1, \ldots, \alpha_m)$ of distinct real numbers is said to be *algebraically dependent* if there is a non-zero polynomial $h(x_1, \ldots, x_m)$ with integer coefficients such that $h(\alpha_1, \ldots, \alpha_m) = 0$. If $\mathcal{A}$ is not algebraically dependent, it is called *generic* [17]. We say that $p = (p_1, \ldots, p_n)$ is generic in 2-space, if its $2n$ coordinates are generic. It can be shown that the set of generic p's form an open dense subset of $\mathbb{R}^{2n}$ [18]. A graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is called *generically rigid*, if $\mathbb{F}(p) = (p, \mathcal{E})$ is rigid (equivalently first-order rigid) for a generic p (and therefore for all generic p, see below).

The property of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}(p)$ but predicts the rigidity of a formation from the graph of the vertices and links, in other words, by the underlying graph. Moreover, this describes the rigidity of $\mathbb{F}(p)$ for almost all configurations p. For this reason, it is a preferred form of the concept of a "rigid formation" for purposes of control. The following theorem summarizes these key properties of a generically rigid graph [16]:

Theorem 2.2: If $|v| > 2$, the following are equivalent:

1) a graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is generically rigid in 2-space;
2) for some p, the formation $\mathbb{F}(p)$ with the underlying graph $\mathbb{G}$ has rank$\{R(\mathbb{F}; p)\}$= $2|\mathcal{V}|-3$ where $|\mathcal{V}|$ denotes the cardinal number of $\mathcal{V}$;
3) for almost all p, the formation $\mathbb{F}(p)$ with the underlying graph $\mathbb{G}$ is rigid.

For 2-space, we have a complete combinatorial characterization of generically rigid graphs, which was first proved by Laman in 1970 [19].

Theorem 2.3 (Laman [19]): A graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is generically rigid in 2-space if and only if there is a subset $\mathcal{E}' \subseteq \mathcal{E}$ satisfying the following two conditions: (1) $|\mathcal{E}'| = 2|\mathcal{V}|-3$, (2) For all $\mathcal{E}'' \subseteq \mathcal{E}', \mathcal{E}'' \neq \emptyset, |\mathcal{E}''| \leq 2|\mathcal{V}(\mathcal{E}'')| - 3$, where $|\mathcal{V}(\mathcal{E}'')|$ is the number of vertices that are end-vertices of the edges in $\mathcal{E}''$.

A generically rigid graph on $|V| \geq 2$ vertices, with $|\mathcal{E}| = 2|\mathcal{V}| - 3$, is a generically minimally rigid graph. We will see, in §VI, that there are efficient algorithms to construct such graphs and to orient generically minimally rigid graphs.

2) Robust Formations: We will be dealing with formations where there is some 'noise' in the link lengths. Accordingly, we need to know whether small changes in the link lengths in a formation $\mathbb{F}(p)$ can be realized with small changes in the underlying configuration p. If, for all small changes in link lengths there is a realization nearby then the formation is *robust*. There is a companion first-order concept which is easier to test (and is equivalent, for almost all configurations, see below). A directed formation $\mathbb{F}(p)$ with lengths $\mathcal{E}$ for the directed edges is first-order robust if the system of equations $R(\mathbb{F};p)X = \delta$ has solutions for all choices of the strains (changes in length) in edge length δ.

When the rows of the rigidity matrix for $\mathbb{F}(p)$ are independent, we say that $\mathbb{F}(p)$ is an *independent formation*, and we also say that the links are *independent*.

Lemma 2.1: A formation $\mathbb{F}(p)$ is first-order robust if and only if the links of $\mathbb{F}(p)$ are independent.

At a generic configuration, the formation $\mathbb{F}(p)$ is first-order robust if and only if it is robust.

Proof: By basic linear algebra, the system $R(\mathbb{F};p)X = \delta$ is solvable for all possible δ if and only if the rows of the matrix are independent. This completes the proof of the first part.

Assume that the formation $\mathbb{F}(p)$ is first-order robust so the rows of the rigidity matrix are independent. Consider the rigidity map: $r_{\mathbb{F}} : \mathbb{R}^{2|\mathcal{V}|} \rightarrow \mathbb{R}^{|\mathcal{E}|}$, which measures the lengths of all links in the formation. The independence of the rows of the rigidity matrix means the Jacobian of $r_{\mathbb{F}}$ (the rigidity matrix) has maximum rank at p, so p is a regular point of this map. As a result, the space of finite motions has the same dimension as the space of first-order motions at the point. Since the first-order map has inverse solutions for all δ (not necessarily unique), it is onto. The map $r_{\mathbb{F}}$ must also be onto in a neighborhood of p. We conclude we can find a solution $q(\delta)$ for each sufficiently small δ, and the formation is robust.

Conversely, assume the formation is robust, and we have a generic configuration. The map $r_{\mathbb{F}}$ is onto. The generic configuration p gives maximum rank to the Jacobian so it is regular for the map $r_{\mathbb{F}}$ and the dimension of the space of first-order motions matches the dimension of the space of motions. Therefore the first-order map is also onto, and the formation is first-order robust. We note that this also means the rows of the rigidity matrix are independent. □

We have seen that, in general, independence of the rows of the rigidity matrix means that $R(\mathbb{F};p) \times \dot{p}^T = \delta$ has solutions for all δ (strains or instantaneous changes in length of bars) where $\dot{p} = (\dot{p}_1, \dot{p}_2, \ldots, \dot{p}_n)$. If $\mathbb{F}(p)$ is not first-order rigid, then there is an affine space of realizations with the given instantaneous strains larger than the space of congruences. First-order rigidity means $R(\mathbb{F};p) \times \dot{p}^T = \delta$ has at most one solution, up to congruence. Independence and first-order rigidity (minimal first-order rigidity) means $R(\mathbb{F};p) \times \dot{p}^T = \delta$ has a unique solution (up to congruence), for all δ. If a point formation is both first-order rigid and first-order independent, the rows form a basis for the possible first-order constraints, and the formation is both minimal first-order rigid and maximal independent among link sets on the given configuration. We call such a formation *first-order basic*. This combination is even better, as a pair of nearby realizations with the same lengths would cause ambiguity and possible vibrations between several possible configurations when faced with noise. We will return to this inherent problem of linear dependence in §V.

Theorem 2.4: For a plane point formation $\mathbb{F}(p)$, the following are equivalent:

1) $\mathbb{F}(p)$ is first-order basic for some configuration;
2) $\mathbb{F}(q)$ is first-order basic for every generic configuration q;
3) $\mathbb{F}(q)$ is robust and rigid for an open dense subset of configurations.

We call a graph *basic* if its corresponding formation satisfies any of these equivalent properties. We will return to this in §V.

3) Sequential Techniques: In this section, we recall sequential techniques to create all basic point formations. As noted earlier, Laman's Theorem characterizes rigidity in 2-space. There are sequential techniques for generating rigid classes of graphs in 2-space based on what are known as the vertex addition and edge splitting. Before explaining these operations and sequences, we introduce some additional terminology. We will omit the discussion of other sequential or inductive techniques here [16], [20], [21].

If (i, j) is an edge, then we say that i and j are *adjacent* or that j is a *neighbor* of i and i is a neighbor of j. The vertices i and j are *incident* with the edge (i, j). Two edges are *adjacent* if they have exactly one common end-vertex. The *degree* or *valency* of a vertex i is the number of neighbors of i. If a vertex has k neighbors, it is called a *vertex of degree* k or a k*-valent vertex*. The set of neighbors of i, denoted by $\mathcal{N}_{\mathbb{G}}(i)$, is called a (open) neighborhood. When i is also included, it is called a closed neighborhood and is denoted by $\mathcal{N}_{\mathbb{G}}[i]$. The subscript $\mathbb{G}$ is usually dropped when there is no danger of confusion.

One operation to extend a generically minimally rigid graph is *vertex addition*: given a generically minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$, we add a new vertex i with two edges between i and two other vertices in $\mathcal{V}^*$ in 2-space. A second operation is *edge splitting*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$, we remove an edge (j, k) in $\mathcal{E}^*$ and then we add a new vertex i with three edges by inserting two edges (i, j), (i, k) and one edge between i and one vertex (other than j, k) in $\mathcal{V}^*$.

Now we are ready to present the following theorems:

Theorem 2.5: (vertex addition in undirected case - Tay, Whiteley [20]) Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ be a graph with a vertex i of degree 2 in 2-space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ denote the subgraph obtained by deleting i and the edges incident with it. Then $\mathbb{G}$ is basic if and only if $\mathbb{G}^*$ is basic.

Theorem 2.6: (edge splitting in undirected case - Tay, Whiteley [20]) Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ be a graph with a vertex i of degree 3, and let $\mathbb{G}' = (\mathcal{V}', \mathcal{E}')$ be the subgraph obtained by deleting i and its three incident edges. Then $\mathbb{G}$ is basic if and only if there is a pair j, k of the neighborhood $\mathcal{N}_{\mathbb{G}}(i)$ such that the graph $\mathbb{G}^* = (\mathcal{V}', \mathcal{E}' \bigcup (j, k))$ is basic.

4) Henneberg Sequences: Henneberg sequences are a systematic way of generating minimally rigid graphs based on the vertex addition and edge splitting operations [20]. In 2-space, a *Henneberg sequence for* $\mathbb{G}$ is a sequence of graphs: $\mathbb{G}_2, \mathbb{G}_3, \ldots, \mathbb{G}_{|\mathcal{V}|} = \mathbb{G}$ such that:

1) $\mathbb{G}_2$ is a single edge with its two vertices;

2) $\mathbb{G}_{i+1}$ comes from $\mathbb{G}_i$ by adding a new vertex either by i) the vertex addition or ii) the edge splitting operation.

Note that $\mathbb{G}_i$ and $\mathbb{G}_{i+1}$ correspond to $\mathbb{G}^*$ and $\mathbb{G}$ in the statements of Theorem 2.5 and Theorem 2.6. All graphs in this sequence are basic rigid in 2-space.

Theorem 2.7 (Henneberg's Theorem [16], [20]): A graph $\mathbb{G}$ with at least two vertices is basic if and only if $\mathbb{G}$ has a Henneberg sequence starting with any preselected edge of $\mathbb{G}$.

If we wanted all generically rigid graphs, we simply add edges to such basic graphs. If we want all generically independent graphs, we simply delete edges from such basic graphs.

III. Rigidity in Directed Formations

In this section, we develop the concept of rigidity for leader-follower (directed) reference formations. This is essentially equivalent to a directed formation of agents under the assumption that agents can perform actions instantly in order to meet their desired distance constraints. This kind of response by agents is similar to the movements that joints perform in a bar-joint framework. Since joints are connected by rigid bars, they continue to satisfy distance constraints instantly when the bar-joint framework is moved. The difference between a directed reference formation and a bar-joint framework is that distance constraints apply on both ends of bars in bar-joint frameworks whereas one end responds to the constraint in directed reference formations.

We will, of course, be building on the results for undirected formations. However, we will also explore directly the requirements for control in directed formations to confirm that what we need is a set of directions satisfying additional properties, whose underlying undirected graph is basic.

First, we give some definitions from graph theory, which are relevant to all directed point formations with leader-follower architecture. A graph in which each edge is replaced by a directed edge is called a *digraph*, also called a *directed graph*. When there is a danger of confusion, we will call a graph, which is not a digraph, an *undirected graph*. A digraph having no multiple edges or loops (corresponding to a binary adjacency matrix with 0's on the diagonal) is called a *simple digraph*.

A directed edge is written with an ordered pair of end-vertices (i, j) representing an edge directed from i to j and drawn with an arrow from i to j, that is from the follower to the leader of that edge. Symmetric pairs of directed edges are called *bidirected edges*. In the context of formations, a bidirected edge is mathematically equivalent to an undirected edge in the underlying graph of a formation, but is effectively a redundant graph with a double edge. In formations that have a leader-follower architecture we will only use digraphs with no bidirected edges.

The number of edges directed into a given vertex i in a digraph $\mathbb{G}$ is called the *in-degree* of the vertex and is denoted by $d^-_{\mathbb{G}}(i)$. The number of edges directed out from a given vertex i in a digraph $\mathbb{G}$ is called the *out-degree* of the vertex and is denoted by $d^+_{\mathbb{G}}(i)$. The out-neighborhood $N^+_{\mathbb{G}}(i)$ of a vertex i is $\{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$, and the in-neighborhood $N^-_{\mathbb{G}}(i)$ of a vertex i is $\{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. The union of out-neighborhood and in-neighborhood is the set of neighbors of i, i.e., the (open) neighborhood of i, $\mathcal{N}_{\mathbb{G}}(i)$. When i is also included, it is the closed neighborhood of i, $\mathcal{N}_{\mathbb{G}}[i]$.

A *directed path* is a sequence $\{i, j, k, \ldots, r, s\}$ such that $(i, j), (j, k), \ldots, (r, s)$ are directed edges of the graph. A *cycle* is a directed path such that the first vertex of the path equals the last. A digraph is *acyclic* if it does not contain any cycle. When stated without any qualification, a cycle of n vertices, denoted by $\mathcal{C}_n$, is assumed to be a simple cycle, meaning every vertex is incident to exactly two edges (that is the vertices of the cycle are distinct). The *length of a cycle* is the number of its edges. Cycles of length 1 are loops. Cycles of length 2 are pairs of multiple edges or equivalently a bidirected edge. We call a cycle of k edges a *k-cycle*. A k-cycle is represented by a $k+1$-tuple of vertices separated by commas, e.g., (i, j, k, i).

A. 2-directed digraphs

Any agent in a leader-follower will be responsible for maintaining the lengths of all of its outgoing links. Since this agent has only two degrees of freedom with which to respond, it can maintain at most two lengths. That is the out-degree can be at most 2. Of course, some agents will be directing the external motion of the formation, so they will reserve some of their freedom to guide the formation, and have out-degree 1, or 0. In summary, the digraph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is 2-*directed*: for all $i \in \mathcal{V}$, $\mathcal{N}^+_{\mathbb{G}}(i) \leq 2$.

A standard type of leader-follower topology is as follows: There is one global leader who does not follow any other agent and one first-follower who only follows the global leader. They are connected with one link pointed from the first-follower to the global leader. The rest of the agents are followers of two other agents. Any agent can also be the leader of other agents. We will call such an architecture a *global leader-first follower* architecture. Fig. 1(a) shows such an example.

From an external control point of view, the leader will direct the translations of the formation, and the first follower will turn to control the rotations of the formation. Since the global leader has no out-going links, the first follower has one link of out-degree 1 and every other agent has out-degree 2, we have $2(n-2)+1 = 2n-3$ links in total. This matches the count needed for the underlying graph to be generically rigid. If any one of the other agents had less than two links, the result would be an additional degree of freedom, and the formation need not be rigid anymore (see below).

One can also consider other types of topologies for leader-follower formations which satisfy the basic rigidity count of $|\mathcal{E}| = 2|\mathcal{V}| - 3$. For example, even with a global leader, the remaining agent of out-degree 1 following only one agent, may not follow the global leader. There will be still one global leader of out-degree 0, one agent of out-degree 1, and other agents of out-degree 2. Fig. 1(b) shows such an example. Another possibility is that all agents have out-degree 2, except three agents of out-degree 1 as shown in Fig. 1(c). We will see, below, that these alternative topologies all contain cycles (Theorem 4.1), while a global leader-first follower architecture may be acyclic.

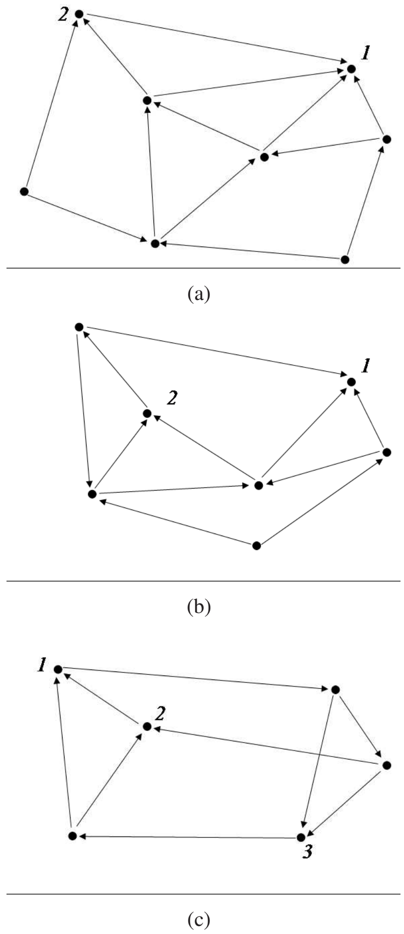


Fig. 1. Three different topologies for a leader-follower architecture are shown. In (a), all vertices are of out-degree 2, except that there is one vertex of out-degree 0 (labeled with 1), and another vertex of out-degree 1 (labeled with 2), and these two vertices are neighbors. In (b), all vertices are of out-degree 2, except that there is one vertex of out-degree 0 (labeled with 1), and another vertex of out-degree 1 (labeled with 2), and these two vertices are not neighbors. In (c), all vertices are of out-degree 2, except that there are three vertices of out-degree 1. These vertices are labeled with 1, 2, and 3.

Even 2-directed digraph with leader first-follower architecture may not have an underlying rigid graph. See the example in Fig. 2. Below, we propose a definition of 'guide rigidity' that will apply if and only if the underlying undirected graph is generically rigid.

B. Guide rigidity for directed formations

We are interested in concepts of motion and guidance under directed constraints. The goal is an intermediate theory of 'rigidity' which respects the direction of links and becomes an appropriate foundation for stable rigidity.

Let $\mathcal{V}_{-2} = \{i \in \mathcal{V} : \mathcal{N}_{\mathbb{G}}^{+}(i) < 2\}$. Given a directed formation $\mathbb{F}$ with an underlying graph $\mathbb{G}$, the *guide freedom* $\Gamma(\mathbb{F})$ is defined by

$$\Gamma(\mathbb{F}) := \sum_{i \in \mathcal{V}_{-2}} \left[2 - \mathcal{N}_{\mathbb{G}}^{+}(i)\right].$$

This guide freedom is the apparent freedom of agents which is unconstrained by the existing leader-follower links of the formation. We will ask that this guide freedom be realized in the possible first-order motions of the formation.

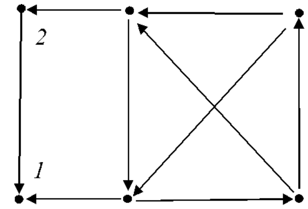


Fig. 2. An example showing that a digraph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ may have an underlying non-rigid graph although it is a 2-directed digraph that has global leader-first follower architecture with count $|\mathcal{E}| = 2|\mathcal{V}| - 3$.

A directed formation $\mathbb{F}(p)$ has *motion control* if the dimension of the space of first order motions tangent to motions of the formation is the guide freedom. A directed formation $\mathbb{F}(p)$ has *first-order motion control* if the dimension of the space of first-order motions is the guide freedom.

Lemma 3.1: If p is generic, then the formation $\mathbb{F}(p)$ has first-order motion control if and only if the formation has motion control.

Moreover, if the rows of the rigidity matrix $R(\mathbb{F}; p)$ are independent, then the formation has motion control if and only if the digraph is 2-directed.

Proof: Assume that the formation is generic. Then the first-order motions form the tangent space to the space of motions in the configuration space with fixed edge lengths. All first-order motions are tangent to motions and the two spaces have the same dimension. The first part is complete.

Assume that the rows are independent. The formation is then generic (or regular) and the formation has control if and only if the formation has first-order motion control. We now show that the formation has first-order motion control, if and only if the digraph is 2-directed.

Let $\Phi(\mathbb{F})$ denote the first-order degrees of freedom of formation $\mathbb{F}(p)$. By the independence, the rank of the rigidity matrix is the number of rows: $|\mathcal{E}| = 2|\mathcal{V}| - \Phi(\mathbb{F})$.

The set of vertices that have out-degree greater than 2 is denoted by $\mathcal{V}_{+2} = \{i \in \mathcal{V} : N_{\mathbb{G}}^{+}(i) > 2\}$. The *excess* $\Omega(\mathbb{F})$ is

$$\Omega(\mathbb{F}) := \sum_{i \in V_{+2}} [N_{\mathbb{G}}^{+}(i) - 2].$$

Since all edges are oriented, the total out-degree is also

$$|\mathcal{E}| = \sum_{i \in \mathcal{V}} N_{\mathbb{G}}^{+}(i) = 2|\mathcal{V}| + \sum_{i \in \mathcal{V}} [N_{\mathbb{G}(i)}^{+} - 2] = 2|\mathcal{V}| - \Gamma(\mathbb{F}) + \Omega(\mathbb{F}).$$

Moreover, by the first-order motion control, $\Gamma(\mathbb{F}) = \Phi(\mathbb{F})$. Together these give the equation:

$$2|\mathcal{V}| - \Phi(\mathbb{F}) = |\mathcal{E}| = 2|\mathcal{V}| - \Gamma(\mathbb{F}) + \Omega(\mathbb{F}).$$

Therefore, $\Omega(\mathbb{F}) = 0$ and $N_{\mathbb{G}}^{+}(i) \leq 2$ for all $i \in \mathcal{V}$.

Conversely, assume the graph is 2-directed, which sets $\Omega(\mathbb{F}) = 0$. Since the edges are independent,

$$2|\mathcal{V}| - \Gamma(\mathbb{F}) = |\mathcal{E}| = 2|\mathcal{V}| - \Phi(\mathbb{F}).$$

Therefore $\Gamma(\mathbb{F}) = \Phi(\mathbb{F})$ and $\mathbb{F}(p)$ has first-order motion control. □

In §2, we presented initial reasons to want independence and rigidity in an undirected formation, in terms of robust rigidity. We now have additional reasons for wanting independence and rigidity in terms of effective control for a directed formation. In terms of simulations, we will reinforce the importance of these properties in the next section.

In the previous subsection, we argued that stable control will require the formation is 2-directed. We say a formation has *effective motion control* if it has motion control and is 2-directed.

Proposition 3.1: A directed formation $\mathbb{F}(p)$ at a generic configuration has effective motion control if and only if it is 2-directed and the links are independent.

Proof: The previous proof shows that if the formation is 2-directed and the links are independent, then the directed formation has effective control.

Assume that the directed formation has effective control. By definition, the graph is 2-directed, and the number of rows is $2|\mathcal{V}| - \Gamma(\mathbb{F})$. If the edges are not independent then $2|\mathcal{V}| - \Phi(\mathbb{F}) = \#(\text{independent rows}) < 2|\mathcal{V}| - \Gamma(\mathbb{F})$. However, since $\mathbb{F}(p)$ has motion control, we also know $\Gamma(\mathbb{F}) = \Phi(\mathbb{F})$. This contradiction shows the edges were independent. □

We summarize the situation both in terms of first-order properties of a single formation and in terms of combinatorial properties of a generic configuration.

Proposition 3.2: For a directed point formation $\mathbb{F}(p)$, the following are equivalent:

1) $\mathbb{F}(p)$ is first-order rigid and has first-order motion control;
2) $\mathbb{F}(p)$ is first-order robust and has first-order motion control;
3) $\mathbb{F}(p)$ has first-order motion control and is 2-directed;
4) $\mathbb{F}(p)$ is first-order robust and is 2-directed;
5) the links of $\mathbb{F}(p)$ are independent and $\mathbb{F}(p)$ is 2-directed.

Proof: The key is that each of the other conditions is equivalent to 4). □

If a directed formation $\mathbb{F}(p)$ has any of these equivalent properties, we say it is *first-order guided*.

Theorem 3.1: At a generic point p, for a directed point formation $\mathbb{F}(p)$, the following are equivalent:

1) $\mathbb{F}(p)$ is first-order guided;
2) the links of $\mathbb{F}(p)$ are independent and $\mathbb{F}(p)$ is 2-directed.
3) $\mathbb{F}(p)$ is robust and has motion control;
4) $\mathbb{F}(p)$ has motion control and is 2-directed;
5) $\mathbb{F}(p)$ is robust and is 2-directed
6) the underlying graph $\mathbb{G}$ is basic and $\mathbb{G}$ is 2-directed.

Proof: For generic configurations, we have already seen that these general properties are equivalent to their first-order equivalents. □

If a directed generic formation $\mathbb{F}(p)$ has any of these equivalent properties, we say the digraph is *guided*. Notice that a digraph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is *guided* if $\mathbb{F}(p) = (p, \mathcal{E})$ is guided (equivalently first-order guided) at all generic configurations p. A directed formation with at least two vertices is *guide rigid* if it is guided and is rigid.

We have analyzed several factors which are relevant to control of directed formations. After working out the connections among these properties, we have provided a single definition which captures all of the essential properties. For generic formations, the properties can be verified by direct algorithms on the underlying digraph. In the next subsection we outline one of these algorithms.

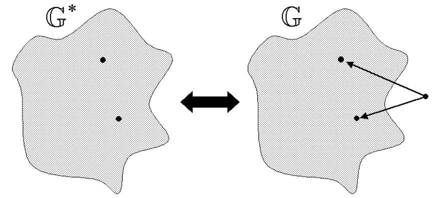


Fig. 3. Vertex Addition - directed case.

C. Sequential Techniques for Generating Guide Rigid Digraphs

As with undirected graphs, one operation for extending a guide rigid graph is *directed vertex addition*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$, we add a new vertex i of out-degree 2 with two edges directed from i to two other vertices in $\mathcal{V}^*$.

Example 3.1: The vertex addition operation for a digraph is shown in Fig. 3.

Theorem 3.2 (vertex addition - directed case): Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ be a digraph with a vertex i of out-degree 2 in 2-space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ denote the subgraph obtained by removing i and the edges incident with it. Then $\mathbb{G}$ is guided (guide rigid) if and only if $\mathbb{G}^*$ is guided (guide rigid).

Proof: Inserting/removing i from the undirected graph $\mathbb{G}$ is equivalent to the vertex addition operation in an undirected graph. Undirected minimally rigid graphs maintain rigidity under the vertex addition operation. Hence generic independence of edges is satisfied in both $\mathbb{G}$ and $\mathbb{G}^*$.

The operation also preserves the generic rigidity of the graph.

Now suppose that $\mathbb{G}$ is 2-directed. If we remove i, then the out-degrees of the vertices of $\mathbb{G}^*$ do not change. Similarly, suppose that $\mathbb{G}^*$ is 2-directed. If we insert i with out-degree 2, then the out-degrees of the remaining vertices do not change and it is still 2-directed. □

The second operation preserving guide rigidity is *directed edge splitting*: given guide rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$, we remove a directed edge (j, k) (directed from j to k) in $\mathcal{E}^*$ and then we add a new vertex i of out-degree 2 and in-degree 1 with three edges by inserting two edges (j, i), (i, k), and one edge between i and one other vertex (other than j, k) in $\mathcal{V}^*$ such that the edge (j, i) is directed from j to i and the other two edges are directed from i to the other vertices.

Example 3.2: The edge splitting operation for a digraph is shown in Figure 4.

Theorem 3.3 (edge splitting - directed case): Let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ be the graph with directed edge (j, k), and let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, be the graph with an added vertex i and added directed edges (i, k), (i, m), (j, i), where $m \in \mathcal{V}^*$. If $\mathbb{G}^*$ is guided (guided rigid), then $\mathbb{G}$ is guided (guided rigid).

Proof: Generic independence, and generic rigidity follows from the edge splitting operation for undirected graphs as

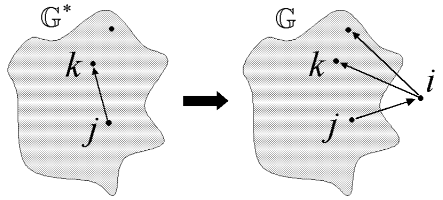
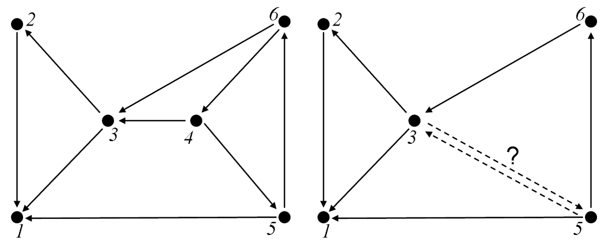
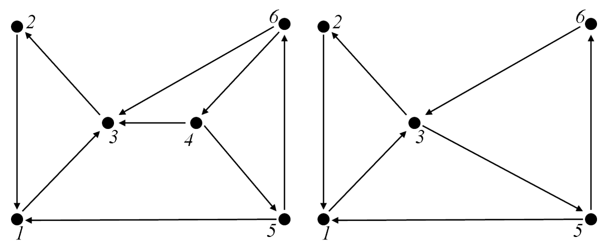


Fig. 4. Edge Splitting - directed case.



(a) A counterexample for the edge splitting operation in the reverse direction.

(b) If vertex 4 is removed, neither $(3, 5)$ nor $(5, 3)$ can be inserted because these edges increase the out-degree of vertices 3 and 5, respectively.



(c) To resolve the difficulty in directed edge splitting, we use edge reversal to make the vertex in question have out degree 1.

(d) The edge $(3, 5)$ can be inserted after edge reversal.

Fig. 5.

explained in §II-B. Suppose that $\mathbb{G}^*$ is 2-directed. The newly inserted vertex i is of out-degree 2. These edges do not change the out-degree of other vertices. In the replacement of the edge (j, k) by (j, i), the out-degree of j is also preserved. □

Remark 3.1: Theorem 3.3 is not in the form of 'if and only if'. A counterexample for reversing the edge splitting operation is shown in Figs. 5(a) and 5(b). There may be some vertex that can be removed in every guide rigid digraph, but it is not every 3-valent vertex that can be removed. Moreover, if we apply these two steps to a single directed edge, we will always have a global leader in the topology. We will see below that there are topologies which meet all our conditions which do no have a global leader. We add one more simple step which will help us generate all guide rigid graphs, with or without a global leader-follower.

The third operation which will complete the constructions is *edge reversal*: edge (a, b) is reversed to (b, a), if for the in-vertex b, $\mathcal{N}^+_{\mathbb{G}}(b) < 2$.

Theorem 3.4 (edge reversal): Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ be a digraph with an edge (a, b); let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ denote the graph obtained by reversing the edge (a, b) to (b, a) if for the in-vertex b, $\mathcal{N}^+_{\mathbb{G}}(b) < 2$. If $\mathbb{G}$ is guided (guide rigid) then $\mathbb{G}^*$ is guided (guide rigid).

Proof: Reversing an edge does not change the underlying undirected graph. Thus generic independence and generic rigidity are maintained under edge reversal. Now suppose that $\mathbb{G}$ is 2-directed. If we reverse (a, b) to (b, a), the only vertex of which out-degree is increased is b. Since $\mathcal{N}^+_{\mathbb{G}}(b) < 2$, then $\mathcal{N}^+_{\mathbb{G}^*}(b) \leq 2$. Therefore $\mathbb{G}^*$ is still 2-directed. □

A *directed Henneberg sequence* for the graph $\mathbb{G}$, is a sequence of steps, starting from a single edge, using the following steps, and ending with the graph $\mathbb{G}$:

1) directed vertex addition;
2) directed edge split;
3) edge reversal.

This final step is related to a key step of the 'pebble game algorithm' [22], [23] used for testing the generic rigidity of graphs in the plane. Throughout the next discussion, we will use the image of 'pebbles' on vertices to visualize the guide freedom from vertices of out-degree less than 2. Overall, in a minimally rigid graph, there will be 3 pebbles (guide freedom 3). The pebbles then can move back down a directed path, as edges are reversed, cascading along, reversing a directed path that previously ran into a vertex with the pebble. When a pebble arrives at a vertex from such a cascade, it now has fewer than two out-directed edges.

Theorem 3.5: These steps all preserve guide rigidity for graphs.

Proof: (i) 2-valent directed addition preserves the property of being minimally rigid, as well as being 2-directed.

(ii) Directed edge splitting preserves the property of being minimally rigid, as well as being 2-directed.

(iii) Edge reversal does not change the graph, and therefore does not change the property of being basic. It also preserves the property of being 2-directed, simply moving the lower out-degree from one vertex to another. □

It is the removal of 3-valent vertices that will require edge reversal. If they have out-degree 2, they may not be removable to proper digraph as shown in Figs. 5(a) and 5(b). To resolve this problem, we make the vertex in question have out degree 1, by reversal, as shown with the example in Fig. 5(c), and then we remove the 3-valent vertex as shown in Fig. 5(d). The following lemma shows that this is always possible.

Lemma 3.2: In a minimally rigid digraph, we can draw a pebble (guide freedom) to any selected vertex a.

Proof: Start at the intended target vertex 0, and search up the digraph, for a free pebble. If we find a free pebble, then it can be drawn down reversing the edges on the path, so that the pebble is at a. We can then take the edge $(0, a)$ and reverse it. If we do not find a free pebble, then the searched subgraph has all vertices of out degree 2, and all out-edges from vertices in this subgraph stay within the subgraph (otherwise we would search more widely). Therefore, the subgraph must have $|\mathcal{E}| = 2|\mathcal{V}|$. This is not possible in a minimally rigid graph which has $|\mathcal{E}| = 2|\mathcal{V}| - 3$. □

Theorem 3.6: All possible guide rigid graphs are generated by a directed Henneberg sequence.

Proof: Assume we have a guide rigid graph. The underlying graph is generically minimally rigid, and is 2-directed.

The result is true for a single edge. Beyond a single ege, it is clear that vertices must have degree at least 2.

Assume the graph is larger with a vertex of degree 2. If this is not of out-degree 2, we do one or two edge reversals to generate an out-degree 2. Then we remove the vertex. The smaller graph $\mathbb{G}^*$ will be basic , guided and 2-directed.

Otherwise, since there are no vertices of degree ≤ 2, by a simple count $\Sigma_{i\in\mathcal{V}}|\mathcal{N}(i)| = 2|\mathcal{E}| = 2(2|\mathcal{V}| - 3)$, the graph must have a vertex 0 of degree 3, connected to vertices a, b, c. By Henneberg's Theorem, this vertex 0 can be removed, and an undirected edge (a, b) for some pair, is inserted to create a new minimally rigid graph $\mathbb{G}^*$, such that $\mathbb{G}^*$ is created from $\mathbb{G}$ by an edge split. It remains to prove that this can be done in a directed manner that preserves the 2-directedness.

There are now three alternatives:
(i) the edge $(a, 0)$ is directed into 0 then $(0, b)$ is directed out (or the reverse). In this case, we can then assign the direction (a, b) to the inserted edge. A directed edge split will return to the original digraph;
(ii) both of the edges $(a, 0)$ and $(b, 0)$ are directed in. Since we have only one out-directed edge, there is a free pebble on 0. This can be used to reverse $(0, b)$ to $(b, 0)$ and we return to case (i);
(iii) both of the edges $(0, a)$ and $(0, b)$ are directed out. The other edge must be directed in. We can use Lemma 3.2 to draw a pebble to 0. This must reverse one of these two key edges. As a result, we return to case (i).

As in the proof of Henneberg's Theorem, we can start with an initial edge, directed in either direction (the direction can be reversed), build up to any generically minimally rigid graph. The previous arguments, show that we can build up to any 2-directed minimally rigid graph, since the necessary edge reversal steps are themselves reversible. □

We now have an inductive construction for all guide rigid digraphs.

IV. Acyclic Formations

In this section, we will refine the results for guide rigidity under the additional assumption that there are no cycles in the leader-follower topology.

Theorem 4.1: For a digraph $\mathbb{G}$, with at least two vertices, the following are equivalent:

1) the digraph is guide rigid and acyclic;
2) the digraph is 2-directed and acyclic, with $|\mathcal{E}| = 2|\mathcal{V}| - 3$;
3) the digraph is constructed from a single edge by a sequence of directed vertex additions;
4) the digraph has global leader-first follower architecture, and is acyclic with all other vertices having out-degree exactly 2.

Proof: (i) implies (ii): We have already seen that guide rigid digraphs are 2-directed and minimally rigid. Minimally rigid digraphs have $|\mathcal{E}| = 2|\mathcal{V}| - 3$;

(ii) implies (iii): We are given that there are no cycles in the digraph. Therefore the digraph represents a partial order between vertices. The partial order can be made into a complete order. The smallest element in the order is the global leader, and the next smallest is the first follower, and this will be our target edge for the directed Henneberg sequence.

Assume there are more than 3 vertices. The maximal element has all edges out so it has degree ≤ 2. Overall, since each edge is directed:

$$\sum_{i\in\mathcal{V}} \mathcal{N}^+(i) = |\mathcal{E}| = 2|\mathcal{V}| - 3.$$

However, the two initial vertices have a total out-degree of 1, so on all other vertices $\mathcal{V}'$:

$$\sum_{i\in\mathcal{V}'} \mathcal{N}^+(i) = |\mathcal{E}| = 2|\mathcal{V}'|.$$

We also know that all vertices have $\mathcal{N}^+(i) \leq 2$, so we conclude that all other vertices have $\mathcal{N}^+(i) = 2$. Take the maximal vertex in the linear order. All edges are out-directed, so the overall valence is 2. We can apply 2-valent vertex removal to get a smaller guide rigid acyclic graph, with the induced linear order. We continue with this until there are only the global leader and first follower. Reversing this sequence gives the desired construction.

(iii) is equivalent to (iv): This is immediate, as the initial edge is the global leader, first-follower edge and the valence assumption is equivalent to the count of (iii).

(iii) implies (i): It is clear from the inductive construction that the digraph is acyclic. From the directed Henneberg sequence it is also clear that the digraph is 2-directed and minimally rigid. Therefore it is guide rigid. □

This result can be easily extended to more general acyclic guided graphs, which are independent but perhaps not rigid, provided we include 'general directed vertex addition' in which the added vertex has valence ≤ 2.

We also note that for formations constructed by vertex addition alone, which engineers call *simple*, we actually know the geometry of which configurations are generic. At each step, we need only to ensure that the added vertex and its two attaching vertices are not collinear [20]. This gives the following geometric corollaries.

Corollary 4.1: A formation $\mathbb{F}(p)$ with an acyclic, global leader-first follower topology, and all other vertices of out-degree exactly two, is first-order guide rigid if and only if for all vertices except the global leader-first follower, the two out-directed edges are not collinear.

Corollary 4.2: Assume that the topology of a directed point formation is acyclic. The underlying digraph is guide rigid if and only if the underlying graph is minimally generically rigid.

Simulation 1 (Acyclic and Guide Rigid): Fig. 6(a) shows a formation created by vertex addition only. Notice that this formation satisfies the condition 3) in Theorem 4.1. The agent with out-degree 0 (global leader) is labeled with 1 and the agent with out-degree 1 is labeled with 2. Agents are fully actuated omnidirectional point agents, i.e., they can move in any direction with any speed. The trajectories of agents obtained in simulations are shown in Fig. 6(b). As the global leader moves on a zigzag trajectory, rigidity is preserved as shown in Fig. 7.

In simulations throughout the paper the following distributed relative distance control is used:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \sum_{j \in \mathcal{N}_{\mathbb{G}}^{+}(i)} \left[d_{ij} - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \right] \begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} \quad (4)$$

for $i \neq 1, 2$ where d_{ij} is the set-point distance between agents i and j. The distances between all agent pairs remain (almost) constant over time as the formation moves. Notice that desired distances are reached asymptotically in (4). So small actuation errors are allowed in simulations, which produce small deviations from constant link lengths in the plots. In simulations throughout the paper, measurement noise levels on link lengths are randomly chosen for each link at the beginning of a simulation program, and remain constant for each link over time during simulation. Noise level ranges between 0% and 15% of link lengths.

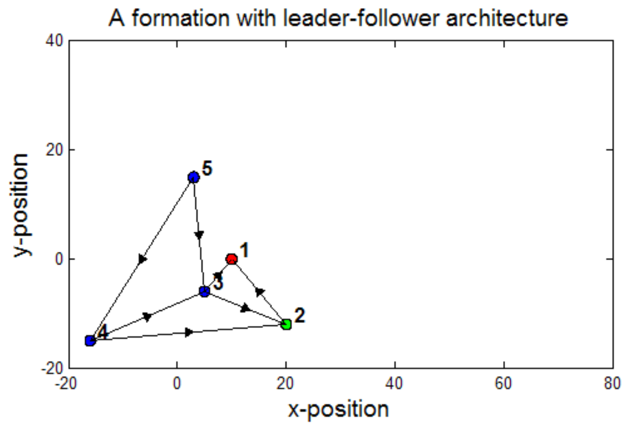
Initial actual distances between agents are set to the desired distances between agents at the beginning of each simulation throughout the paper. But agents have measurement errors. Therefore their measured distances, which are corrupted with noise, do not satisfy the desired distance constraints. Thus they start moving to reach to positions where their measured distances can satisfy the desired distance constraints. While they are doing this, the global leader is also moving on a prescribed zigzag trajectory. Fig. 8 shows a zoomed region of Fig. 7 to show the initial changes in one of the link lengths of the formation shown in Fig. 6(a). Since agents are fully actuated and omnidirectional, transient changes disappear fast. Recall that small actuation errors are allowed in simulations, which produce small deviations from constant link lengths after transient changes disappear in link lengths.

Recall that the global leader and the first follower determine the translation and rotation of the formation. In the simulations throughout the paper, only the global leader's trajectory is prescribed to determine translation. Intentionally, we did not prescribe the one degree of freedom that the first follower had, so that more challenging trajectories are generated to test rigidity. When we prescribe the trajectories of both the global leader and the first follower such that rotations are disallowed, then we obtain translation-only motion.

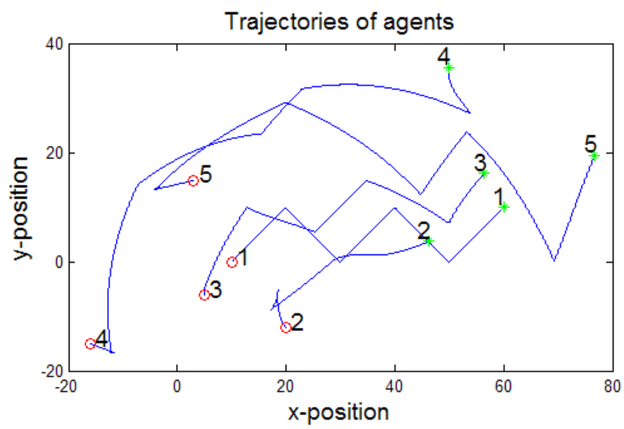
Simulation 2 (Acyclic and Non-Rigid): Fig. 9(a) shows a formation that does not satisfy condition 2) in Theorem 4.1. Although, the underlying digraph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is 2-directed and acyclic, it does not satisfy the edge count, i.e., $|\mathcal{E}| < 2|\mathcal{V}| - 3$. The agent of out-degree 0 (global leader) is colored in red and the agents of out-degree 1 are colored with green. The agents of out-degree 2 are colored in blue. As the global leader moves, the rigidity is lost. This can be seen in Figs. 9(b) and 10. The distances between agent pairs, where there is no sensing/communication link, change over time as the formation moves.

V. Cycles in Directed Formations

The acyclic guide rigid formations have simple constructions and are seen to be stably rigid even for a simple distributed control law of the form in (4). We want to investigate



(a)



(b)

Fig. 6. (a) A rigid formation created by vertex addition only. It satisfies condition 3) given in Theorem 4.1. The agent of out-degree 0 (global leader) is depicted with color red and has index 1. The agent of out-degree 1 is depicted with green and has index 2. The agents of out-degree 2 are depicted with color blue and have indices 3, 4, 5. The set of maintenance links is $\{(2,1),(3,1),(3,2),(4,2),(4,3),(5,3),(5,4)\}$; (b) Trajectories of agents in the formation shown in Fig. 6(a). The initial positions of agents are shown with red circles, and the final positions are shown with green asterisks.

the guide rigid formations which do contain cycles, both via simulations and by direct analysis of possible sources of instability.

It is well known that topologies that contain $\mathcal{C}_2$ cause problems for stability. We note that cycles of length 2 result in redundancies, so they cannot exist in minimally rigid formations or guided formations. The instabilities associated with $\mathcal{C}_2$ were also addressed in [5], and were called information-based instabilities. In [5], [24], problems controlling the external path of a directed formation with $\mathcal{C}_3$ were seen, and the conclusion was that all cycles cause instabilities.

However, our separation of the internal behavior (guide rigidity) from external motion control, lets us resolve the

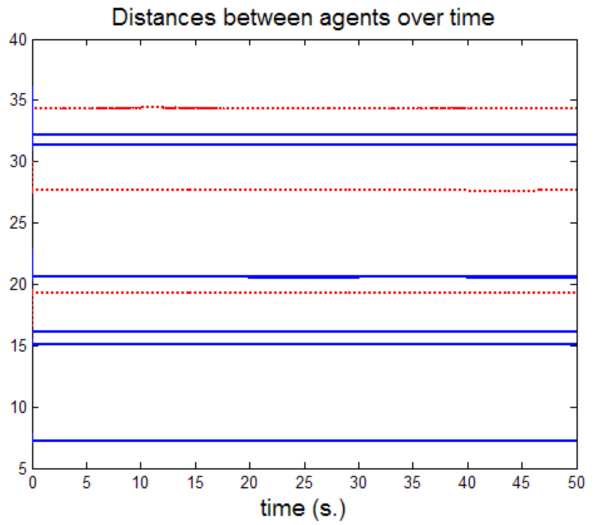


Fig. 7. Distances between agent pairs (of the formation shown in Fig. 6(a)) are shown over time as the global leader moves. The blue solid lines show the distances between agents where there exist links between those pairs. The red dotted lines show the distances between agent pairs where there are no links between those pairs.

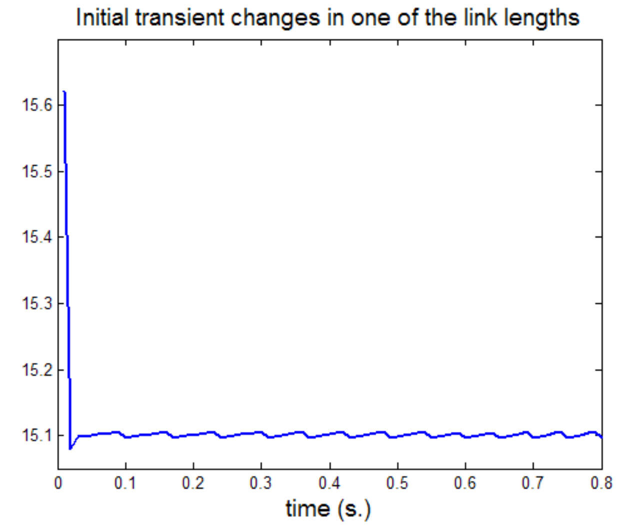


Fig. 8. Initial actual distances between agents are set to the desired distances between agents at the beginning of a simulation. But agents have measurement errors. Therefore their measured distances do not satisfy the desired distance constraints. Thus they start moving to reach to positions where their measured distances (corrupted with noise) can satisfy the desired distance constraints. While they are doing this, the global leader is also moving on a prescribed zigzag trajectory. This figure shows a zoomed region of Fig. 7 to show the initial changes in one of the link lengths of the formation shown in Fig. 6(a). Since agents are fully actuated and omnidirectional, transient changes disappear fast.

problems for such cycles. In this section, we revisit the instabilities associated with the cycles $\mathcal{C}_2$ and $\mathcal{C}_3$. We show that:

- formations with redundant links do have instability, generalizing the analysis of $\mathcal{C}_2$,
- directed formations which are guide rigid, can still be stably rigid even if there are cycles within the formation, generalizing our analysis of $\mathcal{C}_3$.

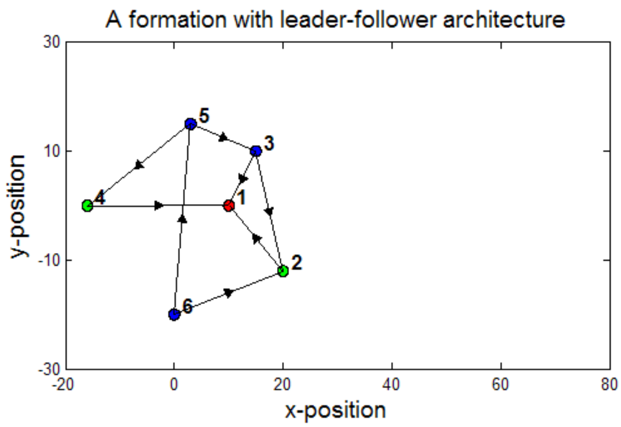
A. The 2-Cycle

In undirected formations, both agents at the end-points of a sensing/communication link maintain a set-distance between each other. For this reason, an undirected link can be considered as two directed links between these two agents with opposite directions, and the underlying graph of an undirected formation can be represented as a directed multigraph where each link is replaced by a cycle of length 2 between the end-points of the link. An example is shown in Fig. 11. We assume that the desired distance between point agent i and point agent j is d_{ij} while the actual distance between these two point agents is $\|p_i - p_j\|$. In real applications, there are measurement errors, for instance due to noise, and it is reasonable to assume that these two agents will have different measurement errors. Let us assume that agent i has a constant measurement error of n_i and agent j has a constant measurement error of n_j. Therefore the measured distance by agent i is $\|p_i - p_j\| + n_i$, and the measured distance by agent j is $\|p_i - p_j\| + n_j$. When agents reach to positions where they satisfy the distance constraint between each other, we would expect that agent i satisfies $\|p_i - p_j\| + n_i = d_{ij}$, and agent j satisfies $\|p_i - p_j\| + n_j = d_{ij}$. We assume that agents act autonomously in a decentralized, non-communicating way. If $n_i \neq n_j$, then there is no way that these two agents will reach positions such that the distance constraint is satisfied by both agents. The agents push and pull each other. Notice that this is a result of inconsistency created by noise in an overdetermined set of equations.

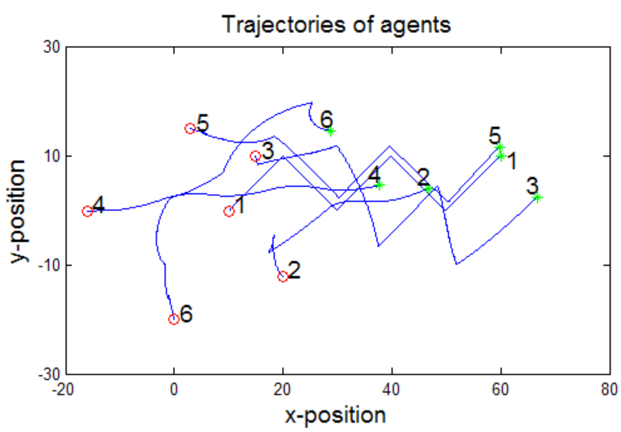
We say that a formation $(\mathcal{V}, \mathcal{E}, f)$ is *realizable* if there exists a point δ such that $r_{\mathbb{F}}(\delta) = f$, using the rigidity map from §III. That is $r_{\mathbb{F}}(\delta) = \|\delta(i) - \delta(j)\| = f(i, j)$. Of course, given any realization, there is a subspace of congruent realizations. If a formation is flexible, it has infinite number of non-congruent realizations beyond the congruent images. If a formation is rigid, it may still have a second non-congruent realization which is flexible. So we do not immediately know whether there are only a finite number of realizations for a given f. However, for a rigid formation, we do know that there is a neighborhood that has only this one realization (up to congruence) [14]. In general, if a graph is generically rigid, then almost all realizations will have at most a finite number of other realizations with the same lengths (up to congruence). If a formation has a unique realization, then it is called *globally rigid*. Beyond a triangle, every generic formation which is globally rigid is redundant.

The underlying graph of a minimally rigid formation, may still have no realization for a given new set of link lengths. This is due to a choice of impermissible link lengths, e.g., the triangle inequality is not satisfied. If a formation is redundantly rigid, a choice of impermissible link lengths, e.g., the triangle inequality, may still be an obstacle to realizing a given f.

Recall that independent links are necessary for effective motion control (Proposition 3.1). There is an additional reason to anticipate that the edges are independent in a formation which can have stable control. In redundantly rigid formations, there is a larger source that usually prevents realization. This



(a)



(b)

Fig. 9. (a) A flexible formation is shown. It does not satisfy condition 2) in Theorem 4.1. Although, the underlying digraph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is 2-directed and acyclic, it does not satisfy the edge count, i.e., $|\mathcal{E}| < 2|\mathcal{V}| - 3$. The agent of out-degree 0 (global leader) is depicted with color red and has index 1. The agents of out-degree 1 are depicted with color green and have indices 2 and 4. The agents of out-degree 2 are depicted with color blue and have indices 3, 5, and 6. The set of maintenance links is $\{(2,1),(3,1),(3,2),(4,1),(5,3),(5,4),(6,2),(6,5)\}$; (b) Trajectories of agents in the formation shown in Fig. 9(a). The initial positions of agents are shown with red circles, and the final positions are shown with green asterisks.

is due to the fact that, in a redundantly rigid formation, the set of equations for link lengths is overdetermined and almost any noise in measurements results in inconsistencies. Inconsistencies in overdetermined systems caused by redundant rigidity are called *redundancy-based inconsistencies*. Thus noisy measurements are a source of redundancy-based inconsistencies that prevent a realization in redundantly rigid formations. To avoid redundancy-based inconsistencies, we focus on guide-rigid, and therefore robust formations which do no have any redundancy.

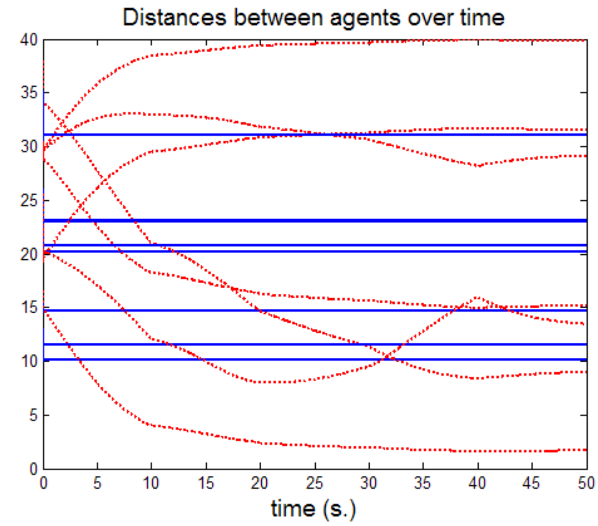


Fig. 10. Distances between agent pairs (of the formation shown in Fig. 9(a)) are shown over time as the global leader moves. The blue solid lines show the distances between agents where there exist links between those pairs. The red dotted lines show the distances between agent pairs where there are no links between those pairs.

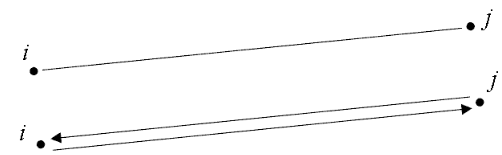


Fig. 11. A link in an undirected formation can be represented by two directed links. Thus it forms a 2-cycle in 1-space. This results in redundant rigidity in 1-space. Measurement errors cause redundancy-based inconsistencies in this overdetermined system.

B. *Analysis of* 3*-cycles*

The behavior of agents on a cycle of length 3 is strikingly different. Let us consider the formation shown in Fig. 12. The underlying graph is minimally rigid. The global leader is labeled with 1, and the first-follower is labeled with 2. Every other vertex has out-degree 2. It can be verified that there is a cycle of length 3 $(4,5,6,4)$. We denote the desired distance between agent i and j by d_{ij}, and write $p_i = (x_i, y_i)$. The position (x_3, y_3) is completely determined by d_{31}, d_{32}, and the positions (x_1, y_1), (x_2, y_2), which are the positions of the global leader and the first follower, respectively. Once the global leader and the first follower take their positions, the agent with index 3 also has a known position. Thus the positions p_1, p_2, p_3 are known. If there are no measurement errors, we expect the following hold: $\|p_i - p_j\| = d_{ij}$, for $(i,j) \in \{(1,4),(2,5),(3,6),(4,5),(5,6),(6,4)\}$. There are six unknowns in this set of equations. If the triangle inequality is satisfied, then we would expect that a solution exists. Consider the case where there are measurement errors denoted by n_{ij} on each link (i,j) on the cycle $(4,6,5,4)$. We expect the following equations hold:

$$\begin{aligned} \|p_i - p_j\| &= d_{ij}, \text{ for } (i,j) \in \{(1,4),(2,5),(3,6)\} \\ \|p_i - p_j\| + n_{ij} &= d_{ij}, \text{ for } (i,j) \in \{(4,5),(5,6),(6,4).\} \end{aligned}$$

There is no reason to expect that introducing n_{ij}'s create an immediate inconsistency as it happened in the case of a 2-cycle. There are no redundant measurements so the system is not overdetermined. Note this is also true for formations that have cycles of length 4 and higher. If, for example, the triangle inequality is not satisfied, then the reason behind it is not cycles themselves, but rather the selection of link lengths. Even acyclic formations with no noise can fail the triangle inequality if link lengths are poorly chosen.

A direct geometric analysis for this formation may also add insight: Consider the case with no noise in measurements in Fig. 13(a). Given the positions p_1, p_2, and p_3 are fixed, then the points p_4, p_5, and p_6 are located such that all six equations are satisfied for the positions shown in Fig. 13(a). Now, let us add noise to the edges $(4,5)$, $(5,6)$, and $(6,4)$ as shown in Fig. 13(b). Clearly, the current positions of point do not satisfy the measured link lengths. The new link lengths determine a unique triangle as shown in Fig. 13(c). Can we locate this triangle such that its vertices touch the three circles but not cross the circles? The answer to this question is 'yes' provided the triangle inequality is satisfied. Therefore there is still a new set of solutions for the positions of points that satisfy the link lengths corrupted with noise. This is shown in Fig. 13(d). For comparison purposes, the solution for the positions of points with no noise are denoted with empty circles in this figure.

A second concern about cycles raised in [24] is the asymptotic instability associated with motions of $\mathcal{C}_3$, as a rigid body, across the plane. This causes an indeterminacy of the positions of the points in $\mathcal{C}_3$. However, this is a result of the absence of a 'tie-down' [20] or adequate motion control for the three guide agents in $\mathcal{C}_3$. The problem is not in the cycle, but rather it is in the lack of constraints to 'direct' the overall motion of the guide rigid formation. It is as if the global leader-first follower agents in our acyclic formation had no fixed position, or direction, and the formation wandered.

Recall that translations and rotations are still allowed in rigid formations. We need to control these 3-degrees of freedom to move a formation sensibly along a target path or towards a target position. This can be done for cycles within a larger architecture, as with the cycle $\mathcal{C}_3$ inside Fig. 12. In a global leader-first follower topology, if the formation is rigid, the global leader and the first follower completely determine the translation and rotation of the formation. The connection to 'cycles' is Theorem 4.1 - all problematic guide agents are in formations with cycles. However, the cycles themselves are not the problem, as we will see below.

Simulation 3 (Guide Rigid with Cycles): Figure 14(a) shows a formation with seven agents. It has three cycles, two of which have length 4, $(7,6,5,3,7)$ and $(4,3,7,6,4)$, and one of which has length 5, $(4,3,7,6,5,4)$. The global leader moves on a zigzag trajectory. The plot of the trajectories of agents are shown in Fig. 14(b). The distances between all agent pairs are shown in Fig. 15.

A final concern about cycles raised in [5] is the situation all point agents move at the same nonzero velocity and are aligned one behind the other in $\mathcal{C}_3$. This formation has a singular geometric configuration which is non-generic and causes a geometry redundance in the rows of the rigidity matrix. This

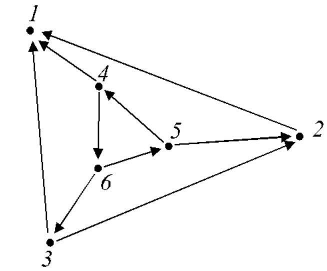


Fig. 12. A point formation that has a 3-cycle, $(4,6,5,4)$. It is minimally rigid in 2-space, and has no redundancy contrary to the redundancy in the formation shown in Fig. 11. Thus measurement errors do not cause redundancy-based inconsistencies in this formation.

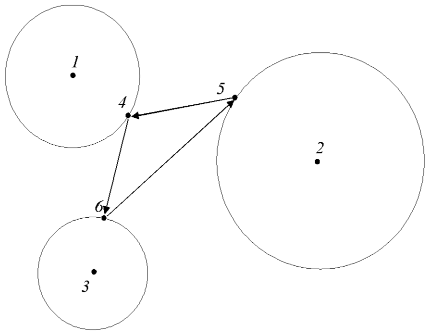
problem is not specific to cycles, but would be a problem even for a global leader-first follower formation of three agents in a triangle on a line. Such geometric redundance does cause instability, as described when we introduced the concept of first-order robust formations. Therefore, we will work with independent formations.

Recall that there are three possibilities for a guide rigid formation: (i) a formation with all agents have out-degree 2 except a global leader of out-degree 0, a first follower of out-degree 1, and these two are connected by a link; (ii) a formation with all agents have out-degree 2 except a global leader of out-degree 0, a free follower of out-degree 1, and these two are not connected by a link; (iii) a formation with all agents of out-degree 2 except three agents of out-degree 1.

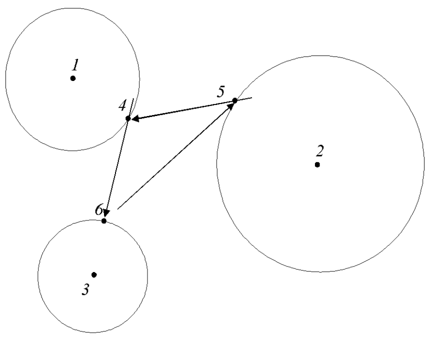
In case (i), with no cycles, we have seen that the formation is stably rigid. If there are cycles, then the proof for directed Henneberg constructions shows that a set of edge reversals on paths will make it a formation which is acyclic, and stably rigid. All that remains is to verify that edge reversal does not change stable rigidity. It is certainly plausible that reversing such a path, while maintaining the same edge lengths, with the same noise, will not alter the realizations or the stability. We conjecture that this is true, but have not provided a detailed argument.

In case (ii), we could add an artificial 'controlling first follower' attached to first global leader, and attach the previous free follower to this vertex. This creates an enlarged guide rigid global leader-first follower architecture. It will have a cycle, since the original formation had a cycle. As long as this new formation is stably rigid, then the only stability problem for the original formation was the 'motion control' information for the original free follower, now captured through the artificial vertex. This could be directly integrated into motion control for this free follower, and the original formation was stably rigid, internally.

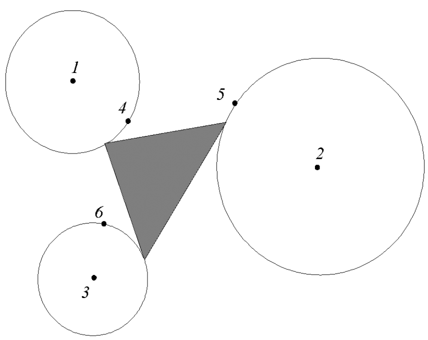
In case (iii), we could add an artificial 'controlling global leader-first follower' pair, and attach the free followers to this pair with three links, two to the global leader and one to the first follower, and create an enlarged guide rigid global leader-first follower architecture. This will contain a cycle, since the original formation had a cycle. As long as this new formation is stably rigid, then the only stability problem for the original formation was the 'motion control' information for the free



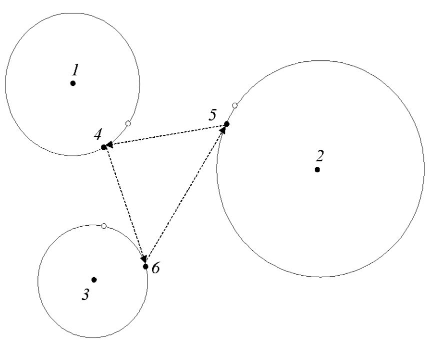
(a)



(b)

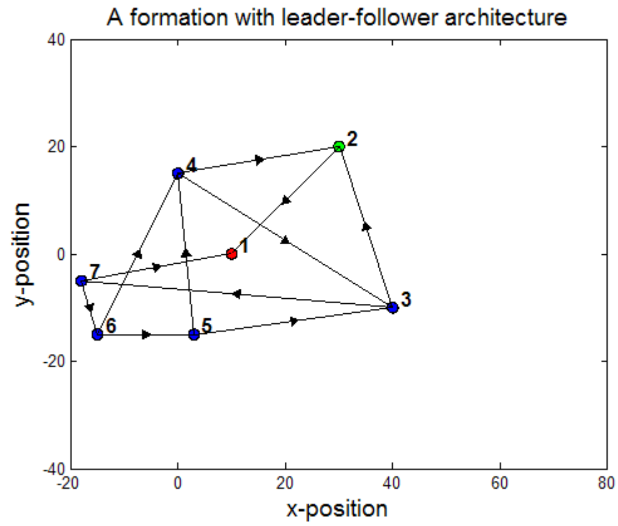


(c)

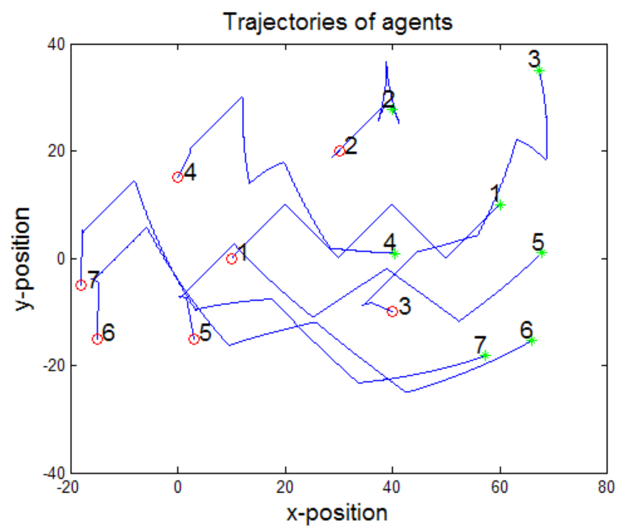


(d)

Fig. 13. (a) This figure shows the distance constraints that need to be satisfied by agents 4, 5, and 6 of the formation shown in Fig. 12. The points in this figure clearly satisfy the constraints; (b) If noise is added to the lengths of links that lie on the 3-cycle $(4, 6, 5, 4)$, then points p_4, p_5, and p_6 fail to satisfy the distance constraints at their current positions; (c) The new distance constraints corrupted with noise can be represented by a triangle. It can be seen that the triangle can be placed between the circles in such a way that its vertices touch the circles at one single point, thus satisfying the distance constraints. (d) The vertices of this triangle determine the new locations of points p_4, p_5, and p_6 as shown with filled circles. The previous positions of the points are shown with empty circles.



(a)



(b)

Fig. 14. (a) A rigid formation created by vertex addition and edge splitting. The agent of out-degree 0 (global leader) is depicted with color red and has index 1. The agent of out-degree 1 is depicted with green and has index 2. The agents of out-degree 2 are depicted with color blue and has indices 3, 4, 5, 6, 7. The set of maintenance links is $\{(2,1),(3,2),(3,7),(4,2),(4,3),(5,3),(5,4),(6,4),(6,5),(7,1),(7,6)\}$. It has three cycles, two of which have length 4, $(7,6,5,3,7)$ and $(4,3,7,6,4)$, and one of which has length 5, $(4,3,7,6,5,4)$; (b) Trajectories of agents in the formation shown in Fig. 14(a). The initial positions of agents are shown with red circles, and the final positions are shown with green asterisks.

followers, now captured through the artificial leader follower. This guidance could be directly integrated into motion control for these free follower, and the original formation was stably rigid, internally.

In summary, we have indicated that the problems of instability can be separated into different forms. Those caused by flexibility are obvious, so we need rigidity. Those caused by redundance (either in the graph or in the geometry) are unavoidable. We need minimal rigidity, and guide rigidity. Once we have guide rigidity, we claim that cycles are not a problem, though they may indicate that the control of the guide agents requires more careful planning.

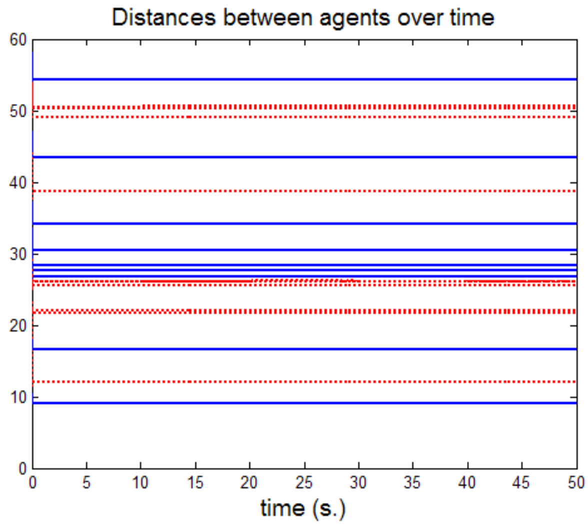


Fig. 15. Distances between agent pairs (of the formation shown in Figure 14(a)) are shown over time as the global leader moves. The blue solid lines show the distances between agents where there exist links between those pairs. The red dotted lines show the distances between agent pairs where there are no links between those pairs.

VI. Creating a Rigid Directed Formation From a Rigid Undirected Formation

Stable rigidity of a directed formation depends not only on the underlying undirected formation but also on the directions of links between agents. In particular, the directed formation must be 2-directed. Given a generically minimally rigid undirected formation, how do we find the directions of links to create a stably rigid directed formation? Below we present one way of doing this.

We start with giving preliminary definitions. A graph is *connected*, if there is a path from any vertex to any other vertex in the graph. A *tree* is a graph in which any two vertices are connected by exactly one path. A *spanning tree* of a connected, undirected graph is a tree which includes every vertex of that graph. There is a standard way of partitioning the edges in a generically minimally rigid graph with the following properties:

1) there are three trees;
2) there are exactly two trees at each vertex;
3) no two non-empty subtrees span the same set of vertices.

These properties define a *3Tree2* partition of the edges [21], [25], [26]. For a generically minimally rigid graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, it is also known that, for each $(i, j) \in \mathcal{E}$, the multigraph obtained by doubling the edge (i, j) is the union of two spanning trees [21], [27].

Now we give a sequential algorithm to find the direction of links to create a stably minimally rigid directed formation from a minimally rigid undirected formation: (Let us assume that i represents the global leader, j represents the first follower connected to i by the edge (j, i).)

Algorithm 6.1: 2-Direction of a Minimally Rigid Formation.

1) Double the edge (j, i) - The entire graph can now be partitioned into two spanning trees.

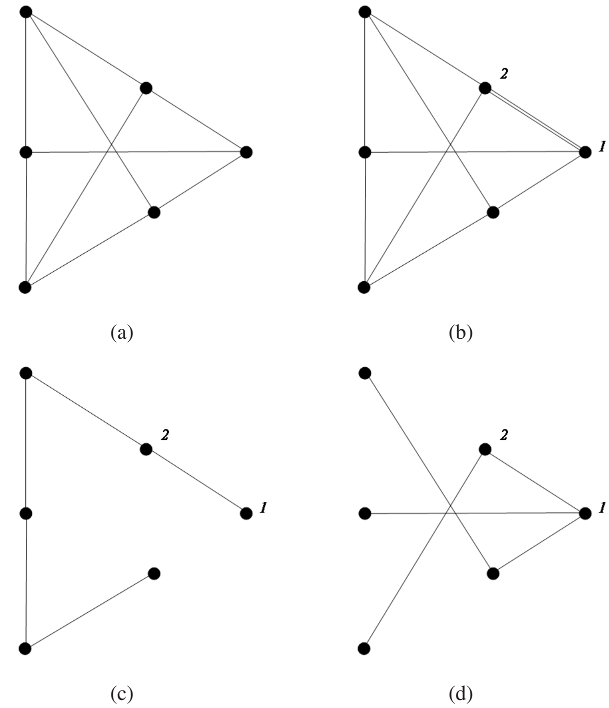


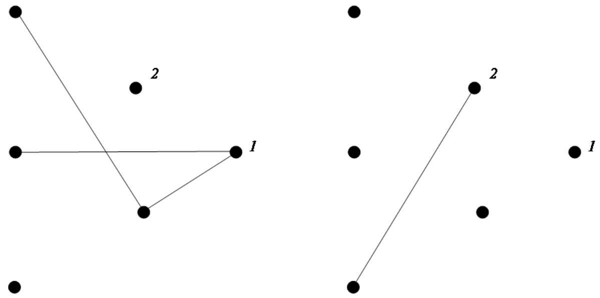
Fig. 16. A minimally rigid point formation is shown in (a). The graph with the double edge $(2, 1)$ is shown in (b). The global leader is labeled with 1 and the first follower is labeled with 2. The graph in (b) can be partitioned into two spanning trees as shown in (c) and (d).

2) Remove (j, i) from one of the two trees - We now have 3-trees, one spanning, and one each containing the original two vertices.
3) Orient the spanning tree down to the selected leader.
4) Orient each of the other two trees down to the global leader or the first follower, whichever is in this revised tree.

This algorithm gives a guide rigid directed formation with out-degree 2 at each point except the first-follower of out-degree 1 and the global leader of out-degree 0. We have conjectured that such a directed formation will always be stably rigid. We give the following example to illustrate this algorithm:

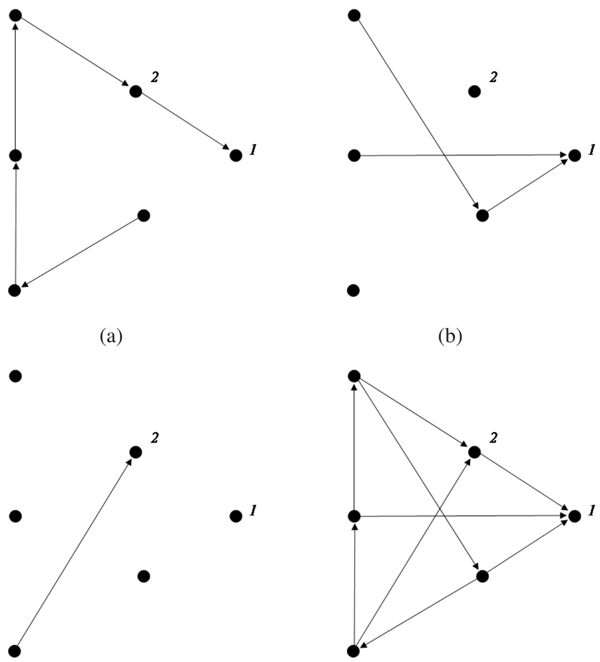
Example 6.1: Consider the generically minimally rigid point formation shown in Figure 16(a). Assume that the global leader is labeled with 1 and the first follower is labeled with 2. The graph with the double edge $(2, 1)$ is shown in Fig. 16(b). This graph can be partitioned into two spanning tress as shown in Figs. 16(c) and 16(d). When we remove $(2, 1)$ from one of the two trees, in this case from Fig. 16(d), we now have three trees: one spanning as shown in Fig. 16(c), and one each containing the original two vertices as shown in Figures 17(a) and 17(b). Fig. 18(a) shows the oriented spanning tree down to the global leader. Figs. 18(b) and 18(c) show the oriented two trees down to the global leader or the first follower. Finally, if we put together the edge topologies in Figs. 18(a), 18(b), and 18(c), we obtain the directed point formation shown in Fig. 18(d).

We note that this algorithm permits an arbitrary choice of the first edge in the graph. There is also a way to deduce this



(a) (b)

Fig. 17. When we remove $(2, 1)$ from one of the two spanning trees in Figs. 16(c) and 16(d), in this case from Fig. 16(d), we now have three trees: one spanning as shown in Fig. 16(c), and one each containing the original two vertices as shown in (a) and (b) in this figure.



(a) (b) (c) (d)

Fig. 18. The oriented spanning tree down to the global leader is shown in (a). The oriented two trees down to the global leader or the first follower are shown in (b) and (c). If we put together the edge topologies in (a), (b) and (c), we obtain the directed point formation shown in (d).

decomposition directly from the assumption that the rigidity matrix has independent rows and full rank [28].

Given a rigid graph $\mathbb{G}$, there is a refined fast (worst case $O(|\mathcal{V}||\mathcal{E}|)$ implemented algorithm (the pebble game) which:

- selects a minimally rigid subgraph in $O(|\mathcal{V}||\mathcal{E}|)$ time and gives an orientation towards a selected leader-follower edge with out-degree 2 on all other vertices for any minimally rigid graph;
- can switch from one such choice of leader-follower edge in a minimally rigid graph to an orientation towards another leader-follower edge in linear time, by cascading pebbles;
- can detect whether there is an acyclic 2-directed orientation towards a given leader-follower edge.

The algorithm is related to our simpler process of finding two spanning trees, as well as the counts in Laman's Theorem. In common with tree finding algorithms, it is greedy, so the order of testing edges etc. does not effect the size of maximal independent sets found, or the distribution of the edges through the agents. The normal implementation can give, as an immediate output, the desired 2-directed graph. Given some set of vertices and independent edges, the algorithm can also select additional edges to extend this to an (oriented) minimally rigid graph, in order $|\mathcal{V}|^2$ time.

There is a third way to generate the digraph. Given a minimally rigid graph, there is a Henneberg sequence starting with the selected global leader-first follower edge. Applied as directed vertex addition and edge splitting, this generates a stably rigid directed formation. Combined with arbitrary cascades of pebbles, these give all possible guide rigid formations. There are order $O(|\mathcal{V}|^2)$ algorithms for directly extracting either a 3Tree2 covering or the Henneberg sequence from the minimally rigid graph. However, some recent implementations for these actually use the pebble game as their core engine.

VII. Concluding Remarks

We provided the necessary and sufficient conditions for guide rigidity in directed reference formations in §III, as a bridge between rigidity for undirected formations, and stable rigidity for directed formations. This analysis addressed problems caused by redundance as well as the possible distribution of directed edges on a minimally rigid digraph. The analysis in §V also confirmed the inconsistencies that result from noisy measurements in redundantly rigid formations. We presented the directed Henneberg constructions as a sequential process for all guide rigid digraphs. We refined those results for acyclic formations §IV, where the guide rigid formations had a simple construction. The analysis in §V confirmed that acyclicity is not a necessary condition for stable rigidity. The cycles are not the real problem, but rather the lack of guide freedom is the reason behind why $\mathcal{C}_3$ has been seen as a problematic topology. Topologies that have cycles within a larger architecture can be stably rigid, and we conjecture that all guide rigid formations (minimally rigid directed formations with all vertices of out-degree at most 2) are stably rigid for internal control. One possible future direction of research is to determine how guide freedom is used to control the otherwise rigid formation.

We anticipate that the pattern of analysis given in this paper will be useful in the analysis of formation rigidity and stability problems and will be a useful tool to create directed rigid formations. We list the overarching comments on two related matters, namely guide rigidity and cycles. Guide rigidity of the formation is lost by:

- the failure of the underlying digraph to be directed rigid;
- redundancy (noise, unrealizability);
- vertices with more than 2 out-degree edges;

Control of the formation is lost, even when the formation itself is guide rigid, if the guide freedom is not handled appropriately (a problem for some alternate topologies). Cycles play only an accidental role in this:

- 2-cycles are redundant, hence problematic;

- alternate topologies (not global leader-first follower) have cycles, so cycles are associated with these problematic situations.

However, we claim cycles inside a global leader-first follower formation are not a problem. Moreover, by adding artificial guide agents as described in §V-B, cycles in the alternate topologies actually occur for subformations within larger global leader-first follower configurations. They are therefore still stably rigid (in the sense of the internal distances). They just require nonstandard contact information.

We want to highlight that this paper strongly suggests that both cyclic and acyclic directed formations can be stably rigid. The question of whether one pattern is 'better' than the other or both types are equivalent in terms of the performance of stable rigidity is still open. It requires more work to determine whether a simple control law, such as the one in (4), is always sufficient, or more sophisticated algorithmic techniques are needed for motion planning so that agents that are part of cycles can reach to their set points. Qualitatively we observe that acyclic formations seem much simpler to work with compared to cyclic formations, but this question requires more quantitative analysis. From a topological point of view, acyclic formations are easier to work with for more decentralized and local operations, such as agent departures, formation splitting and merging. In comparison, cyclic formations can easily get complicated for some of these operations. A quantitative analysis of convergence of agent positions to set points will be useful to realistically compare cyclic and acyclic formations. Although cyclic formations are more difficult to work with at the topological level, they have some advantages over acyclic formations. Acyclicity provides position information flow only in one direction, thus reduces the level of cooperation shared information among agents. If a follower agent in an acyclic formation fails, there is no way that the leaders of that follower can realize this failure. On the other hand, any agent, except the global leader and the first follower, can be a part of a cycle in a rigid formation that has the global leader-first follower architecture. Cycles can provide feedback between agents, and increase coherence among formation members. One possibility for improvement is new topologies that include all agents in cycles. Such topologies will not allow agents of out-degree 0, but will allow agents of out-degree 1. Then there will be a need for consensus among guide agents to direct the translation and rotation of the entire formation sensibly. Such alternate approaches will be explored in the future.

A sequel will give analogs with rigid formations with other types of information structures such as the following: formations with directions, bearings (a.k.a. angle of arrival) in both 2- and 3-space; formations with mixed directions-distances or bearings-distances in 2-space; formations with mixed directed/undirected links. Analogous, significant results have been derived for distance links in 3-space, though the results are only partial for formations with cycles in 3-space. The problems in 3-space are primarily due to the difficulties of characterizing and constructing minimally rigid bar-joint frameworks in 3-space [16], [18], [20].

REFERENCES

[1] C. Belta and V. Kumar, "Abstraction and control for groups of robots," *IEEE Transactions on Robotics*, vol. 20, no. 5, pp. 865–875, 2004.

[2] R. Olfati-Saber and R. Murray, "Graph rigidity and distributed formation stabilization of multi-vehicle systems," in *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, NV, December 2002.

[3] P. Tabuada, G. Pappas, and P. Lima, "Feasible formations of multi-agent systems," *IEEE Transactions on Robotics*, vol. 21, no. 3, pp. 387–392, 2005.

[4] H. G. Tanner, G. J. Pappas, and V. Kumar, "Leader-to-formation stability," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 3, pp. 443–455, June 2004.

[5] J. Baillieul and A. Suri, "Information patterns and hedging Brockett's theorem in controlling vehicle formations," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, December 2003.

[6] F. Zhang, B. Grocholsky, and V. Kumar, "Formations for localization of robot networks," in *Proceedings of the 2004 IEEE International Conference on Robotics and Automation*, April 2004.

[7] W. Ren and R. W. Beard, "A decentralized scheme for spacecraft formation flying via the virtual structure approach," *AIAA Journal of Guidance, Control and Dynamics*, vol. 27, no. 1, pp. 73–82, January 2004.

[8] T. Eren, P. Belhumeur, B. Anderson, and A. Morse, "A framework for maintaining formations based on rigidity," in *Proceedings of the 15th IFAC World Congress, Barcelona, Spain*, July 2002, pp. 2752–2757.

[9] T. Eren, P. Belhumeur, and A. Morse, "Closing ranks in vehicle formations based on rigidity," in *Proceedings of the 41st IEEE Conference on Decision and Control*, December 2002, pp. 2959–2964.

[10] T. Eren, B. D. O. Anderson, A. S. Morse, W. Whiteley, and P. N. Belhumeur, "Operations on rigid formations of autonomous agents," *Communications in Information and Systems*, pp. 223–258, September 2004.

[11] H. G. Tanner, V. Kumar, and G. J. Pappas, "Stability properties of interconnected vehicles," in *15th International Symposium on Mathematical Theory of Networks and Systems*, South Bend, IN, 2002.

[12] T. Eren, W. Whiteley, B. D. O. Anderson, A. S. Morse, and P. N. Belhumeur, "Information structures to secure control of rigid formations with leader-follower architecture," in *Proceedings of the American Control Conference*, Portland, Oregon, June 2005, pp. 2966–2971.

[13] W. Whiteley, "Rigidity and scene analysis," in *Handbook of Discrete and Computational Geometry*, J. Goodman and J. O'Rourke, Eds. CRC Press, 1997, pp. 893–916.

[14] B. Roth and W. Whiteley, "Tensegrity frameworks," *Transactions AMS*, vol. 265, pp. 419–446, 1981.

[15] B. Roth, "Rigid and flexible frameworks," *American Mathematical Monthly*, vol. 88, pp. 6–21, 1981.

[16] W. Whiteley, "Matroids from discrete geometry," in *Matroid Theory*, J. E. Bonin, J. G. Oxley, and B. Servatius, Eds. American Mathematical Society, Contemporary Mathematics, 1996, vol. 197, pp. 171–313.

[17] R. Connelly, "On generic global rigidity," June 2003, preprint, Cornell University, Ithaca, New York, USA.

[18] B. Servatius and H. Servatius, "Generic and abstract rigidity," in *Rigidity Theory and Applications*, M. Thorpe and P. Duxbury, Eds. Kluwer Academic/Plenum Publishers, 1999, pp. 1–19.

[19] G. Laman, "On graphs and rigidity of plane skeletal structures," *Journal of Engineering Mathematics*, vol. 4, pp. 331–340, 2002.

[20] T. Tay and W. Whiteley, "Generating isostatic frameworks," *Structural Topology*, vol. 11, pp. 21–69, 1985.

[21] L. Chavez, L. Moshe, and W. Whiteley, "Bases and circuits for 2-rigidity: constructions via tree partitions," 2004, preprint, York University, Toronto, Ontario, Canada.

[22] B. Hendrickson, "An algorithm for two-dimensional rigidity percolation: The pebble game," *Journal of Computational Physics*, vol. 137, pp. 346–365, 1997.

[23] D. Jacobs and M. Thorpe, "Generic rigidity percolation: The pebble game," *Physical Review Letters*, vol. 75, no. 22, pp. 4051–4054, 1995.

[24] J. Baillieul and A. Suri, "Information patterns and hedging Brockett's theorem in controlling vehicle formations," 2003 CDC Preprint, available from http://iml.bu.edu/CNCS/pub/CDC03.1.pdf.

[25] H. Crapo, "On the generic rigidity of structures in the plane," *Advances in Applied Mathematics*, 1993.

[26] T. Tay, "A new proof of Laman's theorem," *Graphs and Combinatorics*, vol. 9, pp. 365–370, 1993.

[27] A. Recski, “A network approach to the rigidity of skeletal structures. part 2. Laman’s theorem and topological formulae,” *Discrete Applied Mathematics*, vol. 8, pp. 63–68, 1988.
[28] N. White and W. Whiteley, “The algebraic geometry of stresses in frameworks,” *SIAM J. Alg. Disc. Meth.*, vol. 4, pp. 53–70, 1983.