The complexity of indefinite elliptic problems with noisy data

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Abstract. We study the complexity of second-order indefinite elliptic problems $-\operatorname{div}(a\nabla u) + bu = f$ (with homogeneous Dirichlet boundary conditions) over a d-dimensional domain Ω, the error being measured in the $H^1(\Omega)$-norm. The problem elements f belong to the unit ball of $W^{r,p}(\Omega)$, where $p \in [2,\infty]$ and $r > d/p$. Information consists of (possibly-adaptive) noisy evaluations of f, a, or b (or their derivatives). The absolute error in each noisy evaluation is at most δ. We find that the nth minimal radius for this problem is proportional to $n^{-r/d} + \delta$, and that a noisy finite element method with quadrature (FEMQ), which uses only function values, and not derivatives, is a minimal error algorithm. This noisy FEMQ can be efficiently implemented using multigrid techniques. Using these results, we find tight bounds on the ε-complexity (minimal cost of calculating an ε-approximation) for this problem, said bounds depending on the cost $c(\delta)$ of calculating a δ-noisy information value. As an example, if the cost of a δ-noisy evaluation is $c(\delta) = \delta^{-s}$ (for $s > 0$), then the complexity is proportional to $(1/\varepsilon)^{d/r+s}$.

1. Introduction

We study the complexity of elliptic partial differential equations with noisy data. To ease notational burdens, we will consider a class of second-order elliptic problems, although the results in this paper may be extended to include elliptic problems of arbitrary order.

To be specific, we will consider (the variational form of) second-order elliptic problems

$$\begin{aligned} -\operatorname{div}(a\nabla u) + bu &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here, Ω is a fixed d-dimensional domain. We need a specific way to measure the error of an approximation; in this paper, we choose to measure error in the Sobolev H^1-norm, which is equivalent to the problem's natural energy norm. However, the results of this paper also hold if error is measured in the L_2-norm.

Typically, we only know the values of a, b, and f at a finite number of points in Ω. Hence, this is a problem of information-based complexity (IBC) [11]. We cannot obtain exact solutions with finite cost; we must settle for an ε-*approximation*, i.e., an algorithm computing (for each choice of a, b, and f) an approximation whose error is at most ε. For any positive error level ε, we want to know the ε-*complexity* (the minimal cost of calculating an ε-approximation), and we want to

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find algorithms that are optimal, in the sense that they calculate an ε-approximation at (nearly) minimal cost.

Most earlier work on the complexity of elliptic problems has assumed that exact information is available, i.e., we calculate the values these coefficient functions or of the right-hand side exactly. (An extensive treatment of the topic may be found in [12].)

What happens if the information is contaminated by noise? Suppose that we can only guarantee that these calculated function values are accurate to within some positive noise level δ. What can we say about the complexity of such problems?

The systematic study of complexity for problems with noisy information was initiated by [9], with noisy elliptic problems first being investigated in [13]. Let us recall the main results of [13]. If the problem is a *definite* elliptic problem (i.e., a is bounded away from zero and b is non-negative), then the nth minimal radius (the minimal error attained using n evaluations) is $\Theta(n^{-r/d}+\delta)$. This error is attained by a finite element method using quadrature (FEMQ) of degree r (or greater). Now let $c(\delta)$ be the cost of obtaining a δ-noisy approximation of a function value. Then the ε-complexity is

$$\mathrm{comp}(\varepsilon) = \Theta\left(\inf_{0<\delta<C^{-1}\varepsilon}\left\{c(\delta)\left(\frac{1}{C^{-1}\varepsilon-\delta}\right)^{d/r}\right\}\right)$$

for some constant C. A multigrid implementation of the noisy FEMQ, whose noise level δ minimizes the expression above and that uses

$$n = \left\lceil\left(\frac{1}{C^{-1}\varepsilon-\delta}\right)^{d/r}\right\rceil$$

evaluations, is an optimal algorithm. Note that for any $\varepsilon > 0$, the complexity and the choice of the optimal δ depend strongly on the cost function $c(\cdot)$. As a particular case, if $c(\delta) = \Theta(\delta^{-s})$ for some $s > 0$, then the complexity is proportional to $(1/\varepsilon)^{d/r+s}$, and the optimal δ is proportional to ε.

Note that our previous results assumed that the elliptic problem was definite. The problem is said to be *indefinite* if we no longer can assume that the function b is non-negative. Indefinite elliptic problems arise in inverse acoustic scattering theory, in which the Helmholtz problem $-\Delta u+\lambda u = f$ plays an important role (see, e.g., [6]). Here, the only restriction on λ is that it not be an eigenvalue of the Laplacian. Thus, the Helmholtz problem is indefinite.

In this paper, we extend the results of [13] to a class of indefinite elliptic problems. Our main result may be described as follows. Suppose that in (1.1), we have $b(\cdot) = b_0(\cdot) - \lambda$, where the problem

$$\begin{aligned} -\operatorname{div}(a\nabla u) + b_0 u &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

is a definite elliptic problem, and λ is bounded away from the eigenvalues of this problem. (As a particularly important example, the Helmholtz problem satisfies these conditions if λ is bounded away from the spectrum of the Laplacian.) Then the results of [13] still hold. That is, the nth minimal radius and the ε-complexity have the forms given above, and a noisy FEMQ is optimal.

We outline the structure of this paper. In Section 2, we describe the class of elliptic problems to be solved; we also review the basic concepts of IBC of problems for which only noisy information is available. Next, we describe the noisy FEMQ in Section 3. Although the noisy FEMQ used in this paper is the same as that of [13], we give a complete decription here so that this paper will be reasonably self-contained. In Section 4, we show that the noisy FEMQ is a minimal error algorithm, and determine the nth minimal radius of noisy information for our problem.

We describe a multigrid implementation of the noisy FEMQ in Section 5. The multigrid technique used in this paper is almost identical to that in our previous paper [13]. The main difference is that we are not able to use a smoothing operator based on the discretized partial differential operator (DPDO) given by the coefficients a and b, since this operator is not positive definite. Rather, we follow the lead of [3], and use a smoothing operator based on the DPDO given by the coefficients a and 1. Although most of the details of the algorithm are the same as in [13], we give a complete description for the convenience of the reader. Finally, in Section 6, we establish that this multigrid implementation of the noisy FEMQ is an optimal algorithm, and determine the complexity of indefinite elliptic problems for which only noisy information is available.

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2. Problem Description

In what follows, we assume that the reader is familiar with the usual terminology and notations arising in the variational study of elliptic boundary value problems, including Sobolev spaces. See [12, Chapter 5 and Appendix] for further details, as well as the references cited there. For any ordered ring $\mathscr{X}$, let $\mathscr{X}^+$ and $\mathscr{X}^{++}$ respectively denote the nonnegative and strictly positive elements of $\mathscr{X}$, this notation being used mainly when $\mathscr{X} = \mathbb{R}$ or $\mathscr{X} = \mathbb{Z}$. The unit ball of the normed linear space X will be denoted by $\mathscr{B}X$. All O-, Ω-, and Θ-relations will be independent of n, δ, and ε.

We are given $p \in [2, \infty]$, $d \in \mathbb{Z}^{++}$, and $r \in \mathbb{R}$ with $r > d/p$. Let $\Omega \subset \mathbb{R}^d$ be a given bounded, simply-connected region with $\partial\Omega \in C^{r+2}$. We write

$$\langle v, w \rangle = \int_\Omega vw$$

for any $v, w \colon \Omega \to \mathbb{R}$ for which the integral makes sense. For sufficiently smooth $v \colon \overline{\Omega} \to \mathbb{R}$, we define the partial differential operator

$$L_{a,b}v = -\operatorname{div}(a\nabla v) + bv.$$

Here, $a, b \in W^{r,\infty}(\Omega)$. Associated with the operator $L_{a,b}$ is the bilinear form

$$B_{a,b}(v, w) = \int_\Omega [a\nabla v \cdot \nabla w + bvw] \qquad \forall\, v, w \in H_0^1(\Omega),$$

which is obtained when we integrate $\langle L_{a,b}v, w\rangle$ by parts.

Let $M > m > 0$, and let $\zeta, \lambda_0 \in \mathbb{R}^{++}$. We let $\mathscr{A} = \mathscr{A}_{m,M,\lambda_0,\zeta}$ denote the set of all $(a, b) \in [W^{r,\infty}(\Omega)]^2$ satisfying the following conditions:

(1) $a(x) \geq m$ for all $x \in \Omega$.
(2) $b(x) = b_0(x) - \lambda$, where the following conditions hold:
   (a) $b_0(x) \geq 0$ for all $x \in \Omega$.
   (b) $\|a\|_{W^{r,\infty}(\Omega)} \leq M$ and $\|b_0\|_{W^{r,\infty}(\Omega)} \leq M$.
   (c) $|\lambda| \leq \lambda_0$.
   (d) Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of L_{a,b_0}, considered as an unbounded operator in $L_2(\Omega)$. Then

$$\inf_{j \in \mathbb{Z}^{++}} |\lambda_j - \lambda| \geq \zeta.$$

Hence $\mathscr{A}$ is a class of coefficients yielding second-order elliptic problems. The *classical formulation* of such a problem (determined by $(a,b) \in \mathscr{A}$, with $b = b_0 - \lambda$) is to find, for $f\colon \Omega \to \mathbb{R}$, a function $u\colon \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned} L_{a,b}u = -\operatorname{div}(a\nabla v) + (b_0 - \lambda)v = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The *variational formulation* of this problem is to find, for $f \in H^{-1}(\Omega)$, an element $u \in H_0^1(\Omega)$ such that

$$B_{a,b}(u,v) = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega). \tag{2.2}$$

Our class of problem elements will be $F = \mathscr{B}W^{r,p}(\Omega) \times \mathscr{A}$. We define a solution operator $S\colon F \to H_0^1(\Omega)$ by letting $u = S([f;a,b])$ iff u satisfies (2.2), i.e., u is the variational solution to the Dirichlet problem (2.1). Now for any $(a,b) \in \mathscr{A}$, the bilinear form is weakly coercive, symmetric, and bounded (see, e.g., [12, Section 5.2]; a "uniform" proof, independent of $(a,b) \in \mathscr{A}$, is given in Section 4 of this paper). Using the generalized Lax-Milgram Lemma ([1, pg. 112], [8, pg. 310]), it follows that for any $[f;a,b] \in F$, there exists a unique solution $u \in H_0^1(\Omega)$ to (2.2), and so the solution operator S is well-defined.

We wish to calculate approximate solutions to this problem, using *noisy standard information.* To be specific, we will be using uniformly sup-norm-bounded noise. Our notation and terminology is essentially that of [9], although we sometimes use modifications found in [10].

Let $\delta \in [0,1]$ be a *noise level*. For $[f;a,b] \in F$, we calculate *δ-noisy information*

$$N_\delta([f;a,b]) = y = [y_1, \ldots, y_{n(y)}], \tag{2.3}$$

about $[f;a,b]$, where for each index $i \in \{1,\ldots,n(y)\}$, there exist a multi-index $\rho(i)$ and a point $x_i \in \Omega$ such that:

(1) $|\rho(i)| < r - d/p$ and $|y_i - (D^{\rho(i)}f)(x_i)| \le \delta$, or
(2) $|\rho(i)| < r$ and either
  (a) $|y_i - (D^{\rho(i)}a)(x_i)| \le \delta$, or
  (b) $|y_i - (D^{\rho(i)}b)(x_i)| \le \delta$.

Note that N_δ is adaptive information, since all decisions at any stage may depend on the information calculated at previous stages.

Let $\mathbb{N}_\delta([f;a,b])$ denote the set of all such y, i.e., the set of all such noisy information about $[f;a,b]$, and we let $Y = \bigcup_{[f;a,b]\in F} \mathbb{N}_\delta([f;a,b])$ denote the set of all possible noisy information values. Then an *algorithm* using the noisy information $\mathbb{N}_\delta$ is a mapping $\phi\colon Y \to H_0^1(\Omega)$.

We want to solve this problem in the worst case setting. This means that the *cardinality* of information $\mathbb{N}_\delta$ is given by

$$\operatorname{card} \mathbb{N}_\delta = \sup_{y\in Y} n(y),$$

and the *error* of an algorithm ϕ using $\mathbb{N}_\delta$ is given by

$$e(\phi, \mathbb{N}_\delta) = \sup_{[f;a,b]\in F}\ \sup_{y\in\mathbb{N}_\delta([f;a,b])} \|S([f;a,b]) - \phi(y)\|_{H^1(\Omega)}.$$

As usual, we will need to know the minimal error achievable by algorithms using specific information, as well as by algorithms using information of specified cardinality. Let $n \in \mathbb{Z}^+$ and $\delta \in [0,1]$. If $\mathbb{N}_\delta$ is δ-noisy information of cardinality at most n, then

$$r(\mathbb{N}_\delta) = \inf_{\phi \text{ using } \mathbb{N}_\delta} e(\phi, \mathbb{N}_\delta).$$

is the *radius of information*, i.e., the minimal error among all algorithms using given information $\mathbb{N}_\delta$. The nth *minimal radius*

$$r_n(\delta) = \inf\{\, r(\mathbb{N}_\delta) : \operatorname{card} \mathbb{N}_\delta \le n \,\},$$

is the minimal error among all algorithms using noisy information of cardinality at most n. Noisy information $\mathbb{N}_{n,\delta}$ of cardinality n such that

$$r(\mathbb{N}_{n,\delta}) = \Theta\bigl(r_n(\delta)\bigr)$$

is said to be nth *optimal information.* An optimal error algorithm using nth optimal information is said to be an nth *minimal error algorithm.*

Next, we describe our model of computation. We will use the model found in [9, Section 2.9]. Here are the most important features of this model:

(1) For any multi-index ρ, any point $x \in \Omega$, and any function v defined on Ω, the cost of calculating a δ-noisy value of $(D^\rho v)(x)$ is $c(\delta)$. Here, the cost function $c\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing function, with $c(\delta) > 0$ for sufficiently small positive δ.
(2) Real arithmetic operations and comparisons are done exactly, with unit cost.
(3) We are not charged for Boolean operations.
(4) Linear operations over $H_0^1(\Omega)$ are done exactly, with cost $\mathbf{g}$.

For any noisy information $\mathbb{N}_\delta$ and any algorithm ϕ using $\mathbb{N}_\delta$, we shall let $\operatorname{cost}(\phi, \mathbb{N}_\delta)$ denote the worst case cost of calculating $\phi\bigl(\mathbb{N}_\delta([f; a, b])\bigr)$ over all $[f; a, b] \in F$.

Now that we have defined the error and cost of an algorithm, we can finally define the complexity of our problem. We shall say that

$$\operatorname{comp}(\varepsilon) = \inf\{\, \operatorname{cost}(\phi, \mathbb{N}_\delta) : \mathbb{N}_\delta \text{ and } \phi \text{ such that } e(\phi, \mathbb{N}_\delta) \le \varepsilon \,\}$$

is the *ε-complexity* of our problem. An algorithm ϕ using noisy information $\mathbb{N}_\delta$ for which

$$e(\phi, \mathbb{N}_\delta) \le \varepsilon \qquad \text{and} \qquad \operatorname{cost}(\phi, \mathbb{N}_\delta) = \Theta\bigl(\operatorname{comp}(\varepsilon)\bigr)$$

is said to be an *optimal algorithm.*

3. The Noisy FEMQ

In this section, we define the noisy finite element method with quadrature (FEMQ). This is an algorithm using standard information consisting only of function evaluations, i.e., no derivative evaluations are used. Our notation is the standard one found in, e.g., [5] and [12, Chapter 5].

This noisy FEMQ is the same as described in [13]. However, we will give a complete description of the noisy FEMQ, so that this paper can be reasonably self-contained.

We first establish some notation. Let $\hat{K}$ be a fixed polyhedron in $\mathbb{R}^d$. We call $\hat{K}$ a *reference element.* Select a fixed value of $k \in \mathbb{Z}^{++}$, and let $P_k(\hat{K})$ denote the space of polynomials having total degree at most k, considered as functions over $\hat{K}$. We next let K be a (small) *finite element*, i.e., the affine image of $\hat{K}$ under a bijection F_K, where

$$F_K(\hat{x}) = B_K\hat{x} + x_K \qquad \forall\, \hat{x} \in \hat{K}, \tag{3.1}$$

where $B_K \in \mathbb{R}^{d\times d}$ is invertible and $x_K \in \mathbb{R}^d$. Next, we let $\mathscr{T}$ be a triangulation of Ω consisting of finite elements, where each $K \in \mathscr{T}$ is the image of the reference element $\hat{K}$ under the affine bijection F_K. Given this triangulation $\mathscr{T}$, we define a *finite element space*

$$\mathscr{S}(\mathscr{T}) = \bigl\{\, s \in H_0^1(\Omega) : s\big|_K \in P_k(K)\ \forall\, K \in \mathscr{T} \,\bigr\}$$

of *degree* k. We will assume that the following conditions hold:

(1) $\{\mathscr{T}_n\}_{n=1}^{\infty}$ is a family of triangulations of Ω such that $\mathscr{S}_n = \mathscr{S}(\mathscr{T}_n)$ is a finite element space of dimension n.

(2) $\{\mathscr{T}_n\}_{n=1}^{\infty}$ is a *quasi-uniform* family of triangulations, i.e.,

$$\limsup_{n\to\infty} \sup_{K\in\mathscr{T}_n} \frac{h_K}{\rho_K} < \infty,$$

where h_K is the diameter of K and ρ_K is the diameter of the largest sphere contained in K.

(3) Let $\|\cdot\|$ denote the ℓ_2 matrix norm on $\mathbb{R}^d$. Then $\|B_K\| \le 1$ for any element $K \in \mathscr{T}_n$ and any triangulation $\mathscr{T}_n$.

We first recall how the noise-free "pure" FEM is defined. Let $n \in \mathbb{Z}^+$, and let $\{s_1, \ldots, s_n\}$ be a basis for $\mathscr{S}_n$. For $[f; a, b] \in F$, find

$$u_n = \sum_{j=1}^{n} \alpha_j s_j,$$

in $\mathscr{S}_n$ such that

$$B_{a,b}(u_n, s_i) = \langle f, s_i\rangle \qquad (1 \le i \le n). \tag{3.2}$$

If we approximate the integrals appearing in the FEM by numerical quadrature, we get the (noise-free) FEMQ. The quadrature rule used to define the FEMQ is initially defined on the reference element. This reference quadrature rule has the form

$$\hat{I}\hat{v} = \sum_{j=1}^{J} \hat{\omega}_j \hat{v}(\hat{x}_j)$$

for functions $\hat{v}$ defined on $\hat{K}$. Said rule is said to be *exact* of degree q if

$$\int_{\hat{K}} \hat{v} = \hat{I}\hat{v} \qquad \forall\, \hat{v} \in P_q(\hat{K}).$$

We define a local quadrature rule over a particular finite element K as

$$I_K v = \sum_{j=1}^{J} \omega_{j,K}\, v(x_{j,K}),$$

where

$$\omega_{j,K} = \det B_K \cdot \hat{\omega}_j \quad \text{and} \quad x_{j,K} = F_K(\hat{x}_j) \qquad (1 \le j \le J) \tag{3.3}$$

for $K = F_K(\hat{K})$, with F_K given by (3.1). Next, for any $\ell \in \mathbb{Z}^+$, we let

$$\mathscr{N}_\ell = \bigcup_{K\in\mathscr{T}_\ell} \bigcup_{j=1}^{J} \{x_{j,K}\}$$

denote the set of all quadrature nodes in all the elements belonging to $\mathscr{T}_\ell$. This is usually *not* a disjoint union, since a quadrature node on the boundary of one element will be on the boundary of an adjacent element sharing a common face.

Let

$$\nu = \min\{k+1, r\}.$$

We now assume that the following conditions hold:

(1) The smoothness r of the problem elements F satisfies $r \geq 1$ (as well as our previous requirement $r > d/p$).
(2) The degree k of the finite element subspaces $\mathscr{S}_{\tilde{n}}$ satisfies $k > d/p - 1$.
(3) $\hat{I}$ is exact of degree $2k + \nu - 1$ over the reference element $\hat{K}$.

We can now define the noise-free FEMQ. Given $n \in \mathbb{Z}^+$, we define

$$\tilde{n} = \max\{\, \operatorname{card} \mathscr{N}_\ell : \ell \in \mathbb{Z}^+ \text{ and } \operatorname{card} \mathscr{N}_\ell \leq \tfrac{1}{3} n \,\}. \tag{3.4}$$

Roughly speaking, $\tilde{n} = \lfloor \frac{1}{3} n \rfloor$, allowing for the fact that $\tilde{n}$ must be the cardinality of the set $\mathscr{N}_\ell$ of quadrature nodes for some triangulation $\mathscr{T}_\ell$. Let $\{s_1, \dots, s_{\tilde{n}}\}$ denote a basis for the finite element space $\mathscr{S}_{\tilde{n}}$. For $[f; a, b] \in F$, we define a new bilinear form $B_{a,b,\tilde{n}}$ on $\mathscr{S}_{\tilde{n}}$ by

$$\begin{aligned} B_{a,b,\tilde{n}}(v,w) &= \sum_{K \in \mathscr{T}_{\tilde{n}}} \left[\sum_{i,j=1}^{d} I_K(a\, \partial_i v\, \partial_j w) + I_K(bvw) \right] \\ &= \sum_{K \in \mathscr{T}_{\tilde{n}}} \sum_{j=1}^{J} \left[\sum_{i=1}^{d} a(x_{j,K}) \cdot (\partial_i v)(x_{j,K}) \cdot (\partial_i w)(x_{j,K}) + b(x_{j,K}) \cdot v(x_{j,K}) \cdot w(x_{j,K}) \right] \end{aligned}$$

and a linear functional $f_{\tilde{n}}$ on $\mathscr{S}_{\tilde{n}}$ by

$$f_{\tilde{n}}(v) = \sum_{K \in \mathscr{T}_{\tilde{n}}} I_K(fv) = \sum_{K \in \mathscr{T}_{\tilde{n}}} \sum_{j=1}^{J} \omega_{j,K} \cdot f(x_{j,K}) \cdot v(x_{j,K})$$

Then the noise-free FEMQ consists of finding $u_{\tilde{n}}^{\mathrm{Q}} \in H_0^1(\Omega)$ such that

$$B_{a,b,\tilde{n}}(u_{\tilde{n}}^{\mathrm{Q}}, s_i) = f_{\tilde{n}}(s_i) \qquad (1 \leq i \leq n). \tag{3.5}$$

We are ready to define the noisy FEMQ. Given $n \in \mathbb{Z}^+$, we again choose the largest $\tilde{n} \in \mathbb{Z}^+$ satisfying (3.4), and a basis $\{s_1, \dots, s_{\tilde{n}}\}$ for the finite element space $\mathscr{S}_{\tilde{n}}$. We now calculate a noisy version of the information that would be used by the noise-free FEMQ. That is, for each element $K \in \mathscr{T}_{\tilde{n}}$ and each index $j \in \{1, \dots, J\}$, we obtain real numbers $\tilde{a}_{j,K,\delta}$, $\tilde{b}_{j,K,\delta}$ and $\tilde{f}_{j,K,\delta}$ satisfying

$$|\tilde{a}_{j,K,\delta} - a(x_{j,K})| \leq \delta, \tag{3.6}$$
$$|\tilde{b}_{j,K,\delta} - b(x_{j,K})| \leq \delta, \tag{3.7}$$
$$|\tilde{f}_{j,K,\delta} - f(x_{j,K})| \leq \delta. \tag{3.8}$$

Let

$$\tilde{\mathbb{N}}_{n,\delta}([f; a, b]) = \{\tilde{\mathbb{N}}_{n,\delta}(f), \tilde{\mathbb{N}}_{n,\delta}(a), \tilde{\mathbb{N}}_{n,\delta}(b)\}$$

where

$$\begin{aligned} \tilde{\mathbb{N}}_{n,\delta}(a) &= \{\tilde{a}_{j,K,\delta} \text{ satisfying (3.6)} : 1 \leq j \leq J \text{ and } K \in \mathscr{T}_{\tilde{n}} \,\}, \\ \tilde{\mathbb{N}}_{n,\delta}(b) &= \{\tilde{b}_{j,K,\delta} \text{ satisfying (3.7)} : 1 \leq j \leq J \text{ and } K \in \mathscr{T}_{\tilde{n}} \,\}, \\ \tilde{\mathbb{N}}_{n,\delta}(f) &= \{\tilde{f}_{j,K,\delta} \text{ satisfying (3.8)} : 1 \leq j \leq J \text{ and } K \in \mathscr{T}_{\tilde{n}} \,\}, \end{aligned}$$

Clearly, $\tilde{\mathbb{N}}_{n,\delta}$ is noisy information of cardinality at most n. For $[f;a,b] \in F$, we define a new bilinear form $\tilde{B}_{a,b,\tilde{n},\delta}$ on $\mathscr{S}_{\tilde{n}}$ by

$$\tilde{B}_{a,b,\tilde{n},\delta}(v,w) = \sum_{K \in \mathscr{T}_{\tilde{n}}} \sum_{j=1}^{J} \left[\sum_{i=1}^{d} \tilde{a}_{j,K,\delta} \cdot (\partial_i v)(x_{j,K}) \cdot (\partial_i w)(x_{j,K}) + \tilde{b}_{j,K,\delta} \cdot v(x_{j,K}) \cdot w(x_{j,K}) \right]$$

and a linear functional $\tilde{f}_{\tilde{n},\delta}$ on $\mathscr{S}_{\tilde{n}}$ by

$$\tilde{f}_{\tilde{n},\delta}(v) = \sum_{K \in \mathscr{T}_{\tilde{n}}} \sum_{j=1}^{J} \omega_{j,K} \cdot \tilde{f}_{j,K,\delta} \cdot v(x_{j,K}).$$

Then we seek

$$\tilde{u}_{\tilde{n}}^{\mathrm{Q}} = \sum_{j=1}^{\tilde{n}} \alpha_j s_j$$

such that

$$\tilde{B}_{a,b,\tilde{n},\delta}(\tilde{u}_{\tilde{n}}^{\mathrm{Q}}, s_i) = \tilde{f}_{\tilde{n},\delta}(s_i) \qquad (1 \le i \le n). \tag{3.9}$$

The coefficient vector

$$a = [\alpha_1, \ldots, \alpha_n]^T$$

satisfies

$$Ga = b,$$

where

$$G = [\tilde{B}_{a,b,\tilde{n},\delta}(s_i, s_j)]_{1 \le i,j \le \tilde{n}}$$

and

$$b = [\tilde{f}_{\tilde{n},\delta}(s_1), \ldots, \tilde{f}_{\tilde{n},\delta}(s_{\tilde{n}})]^T.$$

We see that $\tilde{u}_{\tilde{n}}^{\mathrm{Q}}$ depends on $[f;a,b]$ only through the noisy information $\tilde{\mathbb{N}}_{n,\delta}([f;a,b])$, and so we write $\tilde{u}_{\tilde{n}}^{\mathrm{Q}} = \tilde{\phi}_{n,\delta}(\tilde{\mathbb{N}}_{n,\delta}([f;a,b]))$, with $\tilde{\phi}_{n,\delta}$ an algorithm using our noisy standard information $\tilde{\mathbb{N}}_{n,\delta}$.

4. The Noisy FEMQ is a Minimal Error Algorithm

In this section, we prove that the noisy FEMQ is well-defined. We also establish an error bound for the noisy FEMQ, which allows us to show that the FEMQ is a minimal error algorithm whenever $k \ge r$.

First, we note that the conditions defining $\mathscr{A}$ imply several important inequalities, which hold independently of $(a,b) \in \mathscr{A}$:

Lemma 4.1.

(1) *Gårding's inequality: There exists a constant $\gamma_0 = \gamma_{0;m} > 0$ such that*

$$B_{a,b}(v,v) \ge \gamma_0 \|v\|^2_{H^1(\Omega)} - \lambda_0 \|v\|^2_{L_2(\Omega)} \qquad \forall\, v \in H^1_0(\Omega),$$

for all $(a,b) \in \mathscr{A}$.

(2) *Weak coercivity of $B_{a,b}$: There exists a constant $\gamma_1 = \gamma_{1;m,M,\lambda_0,\zeta} > 0$ such that*

$$\forall\, v \in H^1_0(\Omega), \exists \text{ nonzero } w \in H^1_0(\Omega) : |B_{a,b}(v,w)| \ge \gamma_1 \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)},$$

for all $(a, b) \in \mathscr{A}$.

(3) *Shift theorem: If* $f \in H^r(\Omega)$, *then there exists a constant* $\sigma = \sigma_{m,M,\lambda_0,\zeta,r}$ *such that*

$$\sigma^{-1}\|S([f; a, b])\|_{H^{r+2}(\Omega)} \leq \|f\|_{H^r(\Omega)} \leq \sigma\|S([f; a, b])\|_{H^{r+2}(\Omega)}, \tag{4.1}$$

for all $(a, b) \in \mathscr{A}$.

(4) *Uniform weak coercivity of* $B_{a,b}$: *There exists a constant* $\gamma = \gamma_{m,M,\lambda_0,\zeta} > 0$ *and an index* $n^* \in \mathbb{Z}^+$ *such that if* $n \geq n^*$, *then*

$$\forall\, v \in \mathscr{S}_{\tilde{n}}, \exists\, w \in \mathscr{S}_{\tilde{n}} : |B_{a,b}(v, w)| \geq \gamma\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}, \tag{4.2}$$

for any $(a, b) \in \mathscr{A}$.

Proof: If we use the inequality

$$B_{a,b}(v, v) \geq m|v|^2_{H^1(\Omega)} - \lambda_0\|v\|^2_{L_2(\Omega)}, \tag{4.3}$$

and the Poincaré inequality $|\cdot|_{H^1(\Omega)} \geq C_\Omega\|\cdot\|_{H^1(\Omega)}$, we see that Gårding's inequality follows, with $\gamma_0 = C_\Omega m$.

We next consider weak coercivity. Let $v \in H^1_0(\Omega)$. Since λ_0 is not an eigenvalue of $L_{a,b}$, there exists a unique $z \in H^3(\Omega)$ satisfying

$$\begin{aligned} L_{a,b}z &= \lambda_0 v \qquad \text{in } \Omega, \\ z &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.4}$$

Then

$$B_{a,b}(v, z) = \lambda_0\|v\|^2_{L_2(\Omega)}. \tag{4.5}$$

Now take $w = v + z$. Using (4.3), (4.5), and the Poincaré inequality, we have

$$B_{a,b}(v, w) \geq m|v|^2_{H^1(\Omega)} \geq \gamma_0\|v\|^2_{H^1(\Omega)}. \tag{4.6}$$

Let $\{z_i\}_{i\in\mathbb{Z}^{++}}$ be an orthonormal $H^1_0(\Omega)$ basis satisfying

$$\begin{aligned} L_{a,b_0}z_i &= \lambda_i z_i \qquad \text{in } \Omega, \\ z &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

If we expand

$$v = \sum_{i=1}^{\infty} \beta_i z_i,$$

we find that

$$z = \sum_{i=1}^{\infty} \frac{\beta_i\lambda_0}{\lambda_i - \lambda} z_i.$$

It now follows that

$$\|z\|_{H^1(\Omega)} \leq \sup_{i\in\mathbb{Z}^{++}} \left|\frac{\lambda_0}{\lambda_i - \lambda}\right| \|v\|_{H^1(\Omega)} \leq \frac{\lambda_0}{\zeta}\|v\|_{H^1(\Omega)}.$$

Thus

$$\|w\|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)} + \|z\|_{H^1(\Omega)} \leq \left(1 + \frac{\lambda_0}{\zeta}\right)\|v\|_{H^1(\Omega)}.$$

Using this inequality along with (4.6), we have

$$B_{a,b}(v,w) \geq \left(\frac{\gamma_0\zeta}{\lambda_0+\zeta}\right)\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)},$$

as required.

We next consider the uniform shift theorem. Let $[f;a,b] \in F$ and let $u = S([f;a,b])$. Using the weak coercivity of $B_{a,b}$, we find that there exists nonzero $w \in H_0^1(\Omega)$ such that

$$B_{a,b}(u,w) \geq \gamma_1\|u\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}.$$

Since $B_{a,b}(u,w) = \langle f, w\rangle_{L_2(\Omega)}$, we find that

$$\gamma_1\|u\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)} \leq B_{a,b}(u,w) \leq \|f\|_{H^{-1}(\Omega)}\|w\|_{H^1(\Omega)},$$

and so

$$\|u\|_{L_2(\Omega)} \leq \|u\|_{H^1(\Omega)} \leq \gamma_1^{-1}\|f\|_{H^{-1}(\Omega)}.$$

From [7, Theorem 8.8], we have the a priori inequality

$$\|u\|_{H^{r+2}(\Omega)} \leq C(\|f\|_{H^r(\Omega)} + \|u\|_{L_2(\Omega)}),$$

the constant C depending only on m, M, λ_0, ζ, and r. Using these last two inequalities, we see that there exists $\sigma' = \sigma'_{m,M,\lambda_0,\zeta,r}$ such that

$$\|u\|_{H^{r+2}(\Omega)} \leq \sigma'\|f\|_{H^r(\Omega)}.$$

The reverse inequality

$$\|f\|_{H^r(\Omega)} \leq \sigma''\|u\|_{H^{r+2}(\Omega)},$$

with $\sigma'' = \sigma''_{m,M,\lambda_0,\zeta,r}$, follows easily from the conditions defining the class $\mathscr{A}$. Combining these last two inequalities, we obtain the uniform shift theorem.

To check uniform weak coercivity, we let $\Pi\colon H_0^1(\Omega) \to \mathscr{S}_{\tilde{n}}$ denote the $\mathscr{S}_{\tilde{n}}$-interpolation operator given by

$$(\Pi v)(x) = \sum_{j=1}^{\tilde{n}} v(x_j)s_j \qquad \forall\, v \in H_0^1(\Omega).$$

Here $x_1,\ldots,x_{\tilde{n}}$ are the interior nodes of the triangulation $\mathscr{T}_{\tilde{n}}$, and $\{s_1,\ldots,s_{\tilde{n}}\}$ is the usual dual finite element basis defined by

$$s_i(x_j) = \delta_{i,j} \qquad (1 \leq i,j \leq \tilde{n}).$$

From the usual finite element approximation theory, there is a positive constant C_1 such that

$$\|z - \Pi z\|_{H^1(\Omega)} \leq C_1\tilde{n}^{-\mu/d}\|z\|_{H^3(\Omega)}$$

for any $z \in H_0^1(\Omega)\cap H^3(\Omega)$. Now choose $(a,b) \in \mathscr{A}$. From the conditions on $\mathscr{A}$, there is another positive constant C_2, depending only on M and λ_0, such that

$$|B_{a,b}(v,w)| \leq C_2\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)} \qquad \forall\, v,w \in H_0^1(\Omega).$$

Let

$$n^* = \left\lceil (C_1C_2\sigma\gamma_1\gamma_0^{-1})^{-d/\mu} \right\rceil.$$

For $v \in \mathscr{S}_{\tilde{n}}$, we let $w = v + \Pi z$, where $z \in H^3(\Omega)$ satisfies (4.4). Following [12, Lemmas 5.2.1 and 5.4.4], we find that w is nonzero and that

$$B_{a,b}(v,w) \geq \frac{\gamma_2}{1+\sigma\lambda_0 + C_1\sigma\lambda_0(n^*)^{-\mu/d}}\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}.$$

Hence the problem is uniform weakly coercive. □

Our main tool is *Strang's Lemma* (see [12, pp. 310–312] for a proof of a version having slightly more restrictive hypotheses).

Lemma 4.2. *Suppose that there exists $\delta_0 \in (0,1]$ and $n^{**} \in \mathbb{Z}^{++}$ such that for any $\delta \in [0,\delta_0]$, any $n \geq n^{**}$ and any $(a,b) \in \mathscr{A}$, we have*

$$|B_{a,b}(v,w) - \tilde{B}_{a,b,\tilde{n},\delta}(v,w)| \leq \tfrac{1}{2}\gamma \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \qquad \forall\, v, w \in \mathscr{S}_{\tilde{n}}, \tag{4.7}$$

where γ is as in (4.2). Then for any $n \geq n^{**}$, any $\delta \in [0,\delta_0]$, and any $[f;a,b] \in F$, there is a unique $\tilde{u}^{\mathrm{Q}}_{\tilde{n}} \in \mathscr{S}_{\tilde{n}}$ such that (3.9) holds. Moreover, there exists a positive constant C, such that if $u = S([f;a,b])$ is the solution to (2.2), then

$$\|u - \tilde{u}^{\mathrm{Q}}_{\tilde{n}}\|_{H^1(\Omega)} \leq C \inf_{v \in \mathscr{S}_{\tilde{n}}} \left[\|u - v\|_{H^1(\Omega)} + \sup_{w \in \mathscr{S}_{\tilde{n}}} \left(\frac{|B_{a,b}(v,w) - \tilde{B}_{a,b,\tilde{n},\delta}(v,w)|}{\|w\|_{H^1(\Omega)}} + \frac{|f(w) - \tilde{f}_{\tilde{n},\delta}(w)|}{\|w\|_{H^1(\Omega)}} \right) \right],$$

the constant C being independent of n, δ, and $[f;a,b]$.

We now recall some preliminary error estimates, whose proofs may be found in [13].

Lemma 4.3. *There exists a positive constant C, depending only on m, M, λ_0, and r, such that*

$$|B_{a,b,\tilde{n}}(v,w) - \tilde{B}_{a,b,\tilde{n},\delta}(v,w)| \leq C\delta \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \qquad \forall\, v, w \in \mathscr{S}_{\tilde{n}}$$

and

$$|f_{\tilde{n}}(v) - \tilde{f}_{\tilde{n},\delta}(v)| \leq C\delta \|v\|_{L_2(\Omega)} \qquad \forall\, v \in \mathscr{S}_{\tilde{n}},$$

for any $\delta > 0$, any $[f;a,b] \in F$, and any $n \geq \tilde{n}$, where $\tilde{n}$ satisfies (3.4). □

We are now ready to prove the main result of this section. Here, and in the remainder of this paper, C will denote a generic constant that depends only on m, M, λ_0, ζ, and r, but whose value may change from place to place.

Theorem 4.1. *There exist $n^{**} \in \mathbb{Z}^{++}$ and $\delta_0 > 0$ such that $\tilde{\phi}_{n,\delta}$ is well-defined for all $n \geq n^{**}$ and all $\delta \in [0,\delta_0]$. Furthermore, there exists a constant C such that*

$$e(\tilde{\phi}_{n,\delta}, \tilde{\mathbb{N}}_{n,\delta}) \leq C(n^{-\mu/d} + \delta),$$

where

$$\mu = \min\{k, r\}.$$

Proof: We first show that $\tilde{\phi}_{n,\delta}$ is well-defined. As in [12, pg. 106], there exists a positive constant C such that

$$|B_{a,b}(v,w) - B_{a,b,\tilde{n}}(v,w)| \leq C n^{-\nu/d} \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \qquad \forall\, v, w \in \mathscr{S}_{\tilde{n}}$$

for any $n \in \mathbb{Z}^{++}$. Using the first inequality in Lemma 4.3, we have

$$|B_{a,b}(v,w) - \tilde{B}_{a,b,\tilde{n},\delta}(v,w)| \leq C(n^{-\nu/d} + \delta) \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \qquad \forall\, v, w \in \mathscr{S}_{\tilde{n}} \tag{4.8}$$

for any $n \in \mathbb{Z}^{++}$ and any $\delta \in [0,1]$. It now follows that there exists $\delta_0 \in (0,1]$ and $n^{**} \in \mathbb{Z}^{++}$ such that (4.7) holds for any $\delta \in [0,\delta_0]$, any $n \geq n^{**}$ and any $(a,b) \in \mathscr{A}$. Thus Strang's Lemma implies that the noisy FEMQ $\tilde{\phi}_{n,\delta}$ is well-defined for any such δ and n.

We now turn to the error of the FEMQ. Let $\delta \in [0,\delta_0]$ and $n \geq n^{**}$. For $[f;a,b] \in F$, let $u = S([f;a,b])$. From [12, pg. 107], there exists $v \in \mathscr{S}_{\tilde{n}}$ such that

$$\|u - v\|_{H^1(\Omega)} \leq C n^{-\mu/d} \|f\|_{H^r(\Omega)} \leq C n^{-\mu/d}, \tag{4.9}$$

the latter since there exists a positive constant C for which

$$\|f\|_{H^r(\Omega)} \le C\|f\|_{W^{r,p}(\Omega)} \le C. \tag{4.10}$$

Now for any $w \in \mathscr{S}_{\tilde{n}}$, we find from [12, pg. 106] that

$$|f(w) - f_{\tilde{n}}(w)| \le C n^{-\nu/d}\|f\|_{H^r(\Omega)}\|w\|_{H^1(\Omega)} \le C n^{-\nu/d}\|w\|_{H^1(\Omega)},$$

where we have again used (4.10). Using this inequality and the second inequality in Lemma 4.3, we have

$$|f(x) - \tilde{f}_{\tilde{n},\delta}(w)| \le C(n^{-\nu/d} + \delta)\|w\|_{H^1(\Omega)}. \qquad (4.11).$$

Use (4.8), (4.11), and (4.9) in Strang's Lemma. Since $\mu \le \nu$, we find

$$\|u - \tilde{u}^Q_{\tilde{n}}\|_{H^1(\Omega)} \le C(n^{-\mu/d} + \delta),$$

as required. □

Using Theorem 4.1, we find

Corollary 4.1.

(1) $r_n(\delta) = \Theta(n^{-r/d} + \delta)$.
(2) *The noisy FEMQ, using a quadrature rule that is exact of degree at least $2k + r - 1$, is a minimal error algorithm if $k \ge r$.*
(3) *Adaption is no stronger than non-adaption.*

Proof: We only need to prove the lower bound

$$r_n(\delta) = \Omega(n^{-r/d} + \delta).$$

But the proof of this bound is exactly the same as the analogous lower bound in [13]. □

5. Multigrid Implementation of the Noisy FEMQ

We have shown that the the FEMQ algorithm $\tilde{\phi}_{n,\delta}$ is an nth minimal error algorithm. Said algorithm consists of three steps. First, we evaluate n information samples (values of f, a, or b). Next, we use these values to construct the $\tilde{n} \times \tilde{n}$ linear system $Ga = b$. So far, the cost of the algorithm is $\Theta(n)$. The final step is solving the linear system $Ga = b$. Unfortunately, we do not know how to solve the linear system in time $\Theta(n)$. For this reason, we will pursue a multigrid implementation of the noisy FEMQ for indefinite ellipitc problems, which is an nth minimal error algorithm whose running time is $\Theta(n)$.

This multigrid method is similar to that in our previous paper [13], the difference being that we use the inner multigrid step of [3] (which has been crafted to work with indefinite problems). Nonetheless, we give a complete description of the method, to help keep this paper reasonably self-contained. As in [13], we use the notation of [4, Chapter 6].

Recall that $\{\mathscr{T}_n\}_{n=1}^{\infty}$ is a quasi-uniform grid sequence. Let us write

$$h_j = \max_{K \in \mathscr{T}_j} h_K$$

for the meshsize of $\mathscr{T}_j$. Recall (from Theorem 4.1) that the noisy FEMQ $\tilde{\phi}_{n,\delta}$ is well-defined if $n \ge n^{**}$. Let

$$n_1 = n^{**} < n_2 < \cdots < n_{l-1} < n_l$$

be a sequence of integers, chosen so that

$$\mathscr{T}_{n_{j-1}} \supset \mathscr{T}_{n_j}, \text{ and thus } \mathscr{S}_{n_{j-1}} \subset \mathscr{S}_{n_j}$$

and

$$h_{n_j} \sim \tfrac{1}{2} h_{n_{j-1}} \qquad (2 \le j \le l). \tag{5.1}$$

We let j be fixed, but arbitrary, index in $\{1, \dots, l\}$. If $p_1, \dots, p_{n_j}$ are the interior nodes of the triangulation $\mathscr{T}_{n_j}$, then we get the standard finite element basis $\{s_1, \dots, s_{n_j}\}$ for $\mathscr{S}_{n_j}$ by requiring that $s_i(p_{i'}) = \delta_{i,i'}$ for $1 \le i, i' \le n_j$ (see, e.g., the discussion in [12, Sections 5.7 and A.2.3]).

We define a mesh-dependent inner product $\langle \cdot, \cdot \rangle_j$ on $\mathscr{S}_{n_j}$ by

$$\langle v, w \rangle_j = h_{n_j}^d \sum_{i=1}^{n_j} v(p_i) w(p_i) \qquad \forall\, v, w \in \mathscr{S}_{n_j}.$$

Then the operator A_j on $\mathscr{S}_{n_j}$ is defined by

$$\langle A_j v, w \rangle_j = \tilde{B}_{a,b,n_j,\delta}(v, w) \qquad \forall\, v, w \in \mathscr{S}_{n_j}.$$

We also define the operator $\hat{A}_j$ on $\mathscr{S}_{n_j}$ by

$$\langle \hat{A}_j v, w \rangle_j = \hat{B}_{a,b,n_j,\delta}(v, w) \qquad \forall\, v, w \in \mathscr{S}_{n_j}.$$

Here, the bilinear form $\hat{B}_{a,b,n_j,\delta}$ on $\mathscr{S}_{n_j}$ is defined to be

$$\hat{B}_{a,b,n_j,\delta}(v, w) = \sum_{K \in \mathscr{T}_{n_j}} \sum_{j=1}^{J} \left[\sum_{i=1}^{d} \tilde{a}_{j,K,\delta} \cdot (\partial_i v)(x_{j,K}) \cdot (\partial_i w)(x_{j,K}) + v(x_{j,K}) \cdot w(x_{j,K}) \right].$$

That is, $\hat{B}_{a,b,n_j,\delta}(v, w)$ approximates

$$\hat{B}_a(v, w) = \int_\Omega [a \nabla v \cdot \nabla w + vw]$$

using numerical quadrature and noisy information about a. Note that $\hat{B}_a$ is a uniformly strongly coercive bilinear form on $H_0^1(\Omega)$, as are the forms $\hat{B}_{a,b,n_j,\delta}$ for $1 \le j \le l$.

Following the proof of [4, Lemma 6.2.8], we obtain an upper bound

$$\rho(\hat{A}_j) \le \Lambda_j = C h_{n_j}^{-2} \tag{5.2}$$

on the spectral radius of $\hat{A}_j$, where the constant C is independent of the index j and the coefficient vector (a, b).

Let us define $f_j \in \mathscr{S}_{n_j}$ by requiring that

$$\langle f_j, s \rangle_j = \tilde{f}_{n_j}(s) \qquad \forall\, s \in \mathscr{S}_{n_j}$$

and let us write $\tilde{u}_j$ for the solution $\tilde{u}_j = \tilde{u}_{n_j}^{\mathrm{Q}}$ of the noisy FEMQ for $\mathscr{S}_{n_j}$, so that

$$A_j \tilde{u}_j = f_j.$$

We then let $I^j_{j-1}\colon \mathscr{S}_{n_{j-1}} \to \mathscr{S}_{n_j}$ be the natural embedding, and let $I^{j-1}_j\colon \mathscr{S}_{n_j} \to \mathscr{S}_{n_{j-1}}$ be its adjoint, i.e.,

$$\langle I^{j-1}_j w, v\rangle_{j-1} = \langle w, I^j_{j-1} v\rangle_j = \langle w, v\rangle_j \qquad \forall\, v \in \mathscr{S}_{n_{j-1}}, w \in \mathscr{S}_{n_j}.$$

Recalling that Λ_j is an upper bound on $\rho(\hat{A}_j)$, we now define the *jth-level multigrid iteration* recursively, in terms of the multigrid iterations at lower levels:

> **function** MG($j\colon \mathbb{Z}^+$; $z_0, g\colon \mathscr{S}_{n_j}$): $\mathscr{S}_{n_j}$;
> **begin**
>   **if** $k = 1$ **then**
>   MG := $A_1^{-1} g$
>   **else**
>   **begin**
> $z_1 := z_0 + \Lambda_j^{-1}(g - A_j z_0)$; { pre-smoothing }
> $\overline{g} := I^{j-1}_j (g - A_j z_1)$; { fine-to-coarse intergrid transfer }
> $q_1 :=$ MG($j-1, 0, \overline{g}$); { error correcting }
> $z_2 := z_1 + I^j_{j-1} q_1$; { coarse-to-fine intergrid transfer }
> $z_3 := z_2 + \Lambda_j^{-1}(g - A_j z_2)$; { post-smoothing }
>   **end**;
>   MG := z_3
> **end**

Then for any index t, the t-fold *full multigrid scheme* produces an approximation $\hat{u}_j$ to $\tilde{u}_j$ as follows:

> **function** FMG($j, t\colon \mathbb{Z}^+$): $\mathscr{S}_{n_j}$;
> **begin**
>   **if** $j = 1$ **then**
> $\hat{u}_j := A_1^{-1} f_1$
>   **else**
>   **begin**
> $u^j_0 := I^j_{j-1} \hat{u}_{j-1}$;
>   **for** $i := 1$ **to** t **do**
> $u^j_i :=$ MG(j, u^j_{i-1}, f_j);
> $\hat{u}_j := u^j_r$
>   **end**;
>   FMG := $\hat{u}_j$
> **end**

Let

$$\overline{\mathbb{N}}_{n,\delta} = [\tilde{\mathbb{N}}_{n_1,\delta}, \tilde{\mathbb{N}}_{n_2,\delta}, \ldots, \tilde{\mathbb{N}}_{n_l,\delta}],$$

with l the maximal index for which card $\overline{\mathbb{N}}_{n,\delta} \le n$. Then we may write

$$\hat{u}_l = \overline{\phi}_{n,\delta}\big(\overline{\mathbb{N}}_{n,\delta}([f; a, b])\big),$$

where $\overline{\phi}_{n,\delta}$ is the *full multigrid algorithm.*

Note that the main difference between this multigrid scheme and the one used in [13] is that we use different smoothing operators.

The main result for this section is

THEOREM 5.1.

(1) *The full multigrid algorithm is well-defined.*
(2) *There exits an index t such that the error of the full multigrid algorithm is*

$$e(\overline{\phi}_{n,\delta}, \overline{\mathbb{N}}_{n,\delta}) = O(n^{-\mu/d}),$$

where (as in Theorem 4.1)

$$\mu = \min\{k, r\}.$$

(3) *The combinatory cost of the full multigrid scheme* FMG(l,t) *is* $\Theta(n)$.

Here, the O- and Θ- constants depend only on m, M, λ_0, ζ, and r.

PROOF: The well-definedness follows from Theorem 4.1. To prove the desired error estimate, let us first consider the jth-level multigrid iteration. Let $\|\cdot\|_{E_j}$ be the energy norm defined by

$$\|v\|_{E_j} = \hat{B}_{a,b,n_j,\delta}(v,v)^{1/2},$$

this energy norm being equivalent to the usual $H_0^1(\Omega)$-norm, the equivalency constants depending only on m, M, λ_0, ζ, and r. Since the pre-smoothing and post-smoothing operators for the jth-level multigrid step satisfy the hypotheses of [3], it follows that there exists a constant $\theta \in (0,1)$ such that

$$\|z - \mathrm{MG}(j, z_0, g)\|_{E_j} \leq \theta \|z - z_0\|_{E_j}, \tag{5.3}$$

the constant θ being independent of $g, z, z_0 \in \mathscr{S}_{n_j}$, $j \in \mathbb{Z}^+$, and $[f; a, b] \in F$.

Since a jth-level multigrid iteration reduces the error by a constant factor θ, the error estimate for the full multigrid algorithm follows, exactly as shown in [13, Theorem 5.1].

To determine the combinatory cost of the full multigrid scheme, we first determine the cost of the jth-level scheme. As in [13, Theorem 5.1], we find that the cost of the jth-level scheme is $O(n_j)$, and that the cost of the full multigrid scheme is $O(n)$. □

6. COMPLEXITY

In this Section, we determine the complexity of the noisy indefinite elliptic problem.

It will be useful to explicitly specify some of the order-of-magnitude constants in some of the estimates in the previous sections. Thus, Corollary 4.1 tells us that there exists a positive constant C_1 such that

$$r_n(\delta) \geq C_1(n^{-r/d} + \delta). \tag{6.1}$$

Moreover, let $\tilde{\phi}_{n,\delta}$ be the noisy FEMQ of degree $k \geq r$, using a quadrature rule that is exact of degree at least $2k + r - 1$. Then by Theorem 5.1, there exist positive constants C_2 and $C_3 = C_3(\mathbf{g})$ such that

$$e(\overline{\phi}_{n,\delta}, \overline{\mathbb{N}}_{n,\delta}) \leq C_2(n^{-r/d} + \delta) \tag{6.2}$$

and

$$\mathrm{cost}(\overline{\phi}_{n,\delta}, \overline{\mathbb{N}}_{n,\delta}) \leq C_3 c(\delta) n. \tag{6.3}$$

As in [13, Theorem 7.1], we have

THEOREM 6.1. *The problem complexity is bounded from below by*

$$\operatorname{comp}(\varepsilon) \geq \inf_{\delta>0} \left\{ c(\delta) \left\lceil \left(\frac{1}{C_1^{-1}\varepsilon - \delta} \right)^{d/r} \right\rceil \right\}, \tag{6.4}$$

and from above by

$$\operatorname{comp}(\varepsilon) \leq C_3 \inf_{\delta>0} \left\{ c(\delta) \left\lceil \left(\frac{1}{C_2^{-1}\varepsilon - \delta} \right)^{d/r} \right\rceil \right\}. \tag{6.5}$$

The upper bound is attained by using the noisy FEMQ $\overline{\phi}_{n,\delta}$ described above, with

$$n = \left\lceil \left(\frac{1}{C_2^{-1}\varepsilon - \delta} \right)^{d/r} \right\rceil, \tag{6.6}$$

and with δ chosen minimizing (6.5). □

Hence

$$\operatorname{comp}(\varepsilon) = \Theta\left(\inf_{\delta>0} \left\{ c(\delta) \left(\frac{1}{C^{-1}\varepsilon - \delta} \right)^{d/r} \right\} \right), \tag{6.7}$$

for some constant C. This allows us to determine the complexity for various cost functions $c(\cdot)$, as well as to find the optimal noise level δ for a particular value of ε.

For example, suppose that $c(\delta) = \delta^{-s}$, where $s > 0$. Then for any $\varepsilon > 0$, the optimal δ is

$$\delta^* = \frac{rs\varepsilon}{C(rs+d)} = \Theta(\varepsilon),$$

so that

$$\operatorname{comp}(\varepsilon) = \Theta\left(\left(\frac{d}{sr} \right)^s \left(\frac{C(rs+d)}{d\varepsilon} \right)^{d/r+s} \right) = \Theta\left(\left(\frac{1}{\varepsilon} \right)^{d/r+s} \right).$$

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