The complexity of multivariate elliptic problems with analytic data

Technical Report CUCS-016-94

Arthur G. Werschulz

Division of Science and Mathematics, Fordham University
College at Lincoln Center
New York, NY 10023
and
Department of Computer Science
Columbia University
New York, NY 10023

June 28, 1994

Abstract. Let F be a class of functions defined on a d-dimensional domain. Our task is to compute H^m-norm ε-approximations to solutions of $2m$th-order elliptic boundary-value problems $Lu = f$ for a fixed L and for $f \in F$. We assume that the only information we can compute about $f \in F$ is the value of a finite number of continuous linear functionals of f, each evaluation having cost $c(d)$. Previous work has assumed that F was the unit ball of a Sobolev space H^r of fixed smoothness r, and it was found that the complexity of computing an ε-approximation was $\mathrm{comp}(\varepsilon, d) = \Theta(c(d)(1/\varepsilon)^{d/(r+m)})$. Since the exponent of $1/\varepsilon$ depends on d, we see that the problem is intractable in $1/\varepsilon$ for any such F of fixed smoothness r. In this paper, we ask whether we can break intractability by letting F be the unit ball of a space of infinite smoothness. To be specific, we let F be the unit ball of a Hardy space of analytic functions defined over a complex d-dimensional ball of radius greater than one. We then show that the problem is tractable in $1/\varepsilon$. More precisely, we prove that $\mathrm{comp}(\varepsilon, d) = \Theta(c(d)(\ln 1/\varepsilon)^d)$, where the Θ-constant depends on d. Since for any $p > 0$, there is a function $K(\cdot)$ such that $\mathrm{comp}(\varepsilon, d) \le c(d)K(d)(1/\varepsilon)^p$ for sufficiently small ε, we see that the problem is tractable, with (minimal) exponent 0. Furthermore, we show how to construct a finite element p-method (in the sense of Babuška) that can compute an ε-approximation with cost $\Theta(c(d)(\ln 1/\varepsilon)^d)$. Hence this finite element method is a nearly optimal complexity algorithm for d-dimensional elliptic problems with analytic data.

1. Introduction

We are interested in the approximate solution of linear elliptic boundary-value problems $Lu = f$, where L is a fixed linear elliptic operator of order $2m$ and f belongs to a class F of problem elements. For most problems that arise in practice, F is a space of functions defined on a d-dimensional domain.

Most work on the computational complexity of such problems has assumed that F has been the unit ball of a Sobolev space H^r, see, e.g., [13]. Suppose that the only information we have about any $f \in F$ is the values of a finite number of continuous linear functionals of f, and that the cost of each of these evaluations is $c(d)$. Let us further suppose that we

This research was supported in part by the National Science Foundation under Grant CCR-91-01149.

measure error in the H^m-norm.[1] Then for such classes F, we find that the ε-complexity in the worst case setting satisfies $\mathrm{comp}(\varepsilon, d) = \Theta\big(c(d)\varepsilon^{-d/(r+m)}\big)$.

Suppose we wish to solve elliptic problems for which the dimension d is large. (Examples of such problems include high-dimensional random walks and simultaneous Brownian motion of many non-interacting particles.) If the smoothness r does not increase along with d, we find that $\mathrm{comp}(\varepsilon, d)$ grows faster than $\Theta\big(c(d)(1/\varepsilon)^p\big)$ for any fixed p as $\varepsilon \to 0$. Since faster-than-polynomial growth is the hallmark of intractability (see, e.g., [6]), this means that elliptic boundary-value problems are intractable in the worst case setting for large dimension d, whenever the class F is the unit ball of a Sobolev space of fixed smoothness r. Hence, if we wish to break the inherent intractability of this problem for the worst case setting, we cannot use these balls of fixed smoothness as our class of problem elements.

One idea is to use functions f of infinite smoothness, rather than of fixed smoothness. Perhaps the most natural class to consider would be a class of analytic functions. This idea was first studied in [14] and [15], where we saw that one-dimensional elliptic problems are far easier to solve when the problem elements are a class of analytic (but not piecewise-analytic) functions instead of a class of functions with fixed smoothness. In this paper, we pursue this idea, asking whether d-dimensional elliptic problems are tractable when the problem inputs are analytic.

The main result of this paper is that elliptic boundary-value problems with analytic data are tractable in $1/\varepsilon$, if F is the unit ball of H^∞, a Hardy space of analytic functions. More precisely, we show that $\mathrm{comp}(\varepsilon, d) = \Theta\big(c(d)(\ln 1/\varepsilon)^d\big)$, the Θ-constant possibly depending on d. From this result, it then follows that for any $p > 0$, there is a function $K(\cdot)$ and an $\varepsilon_0 > 0$ such that $\mathrm{comp}(\varepsilon, d) \le c(d)K(d)(1/\varepsilon)^p$ for $0 < \varepsilon < \varepsilon_0$. Hence the problem is tractable in $1/\varepsilon$, with (minimal) exponent 0. Furthermore, there is a finite element p-method[2] whose cost is $\Theta\big(c(d)(\ln 1/\varepsilon)^d\big)$. Hence this finite element method can compute an ε-approximation with cost that is optimal (to within a multiplicative factor depending only on d), i.e., this method is a nearly optimal complexity algorithm.

Note that we allow the evaluation of any continuous linear functional of f as information about f. However, the nearly optimal complexity algorithm is a finite element method using only function evaluations. Hence, standard information consisting of function values is as powerful as arbitrary continuous linear information for our problem. This result should be contrasted to the results for spaces of fixed smoothness reported in [13], in which we found that there was a heavy penalty associated with using function values instead of more general continuous linear functionals.

[1] We choose the H^m-norm mainly for the sake of specificity. However, the usual reason for looking at error estimates in this norm is that it is equivalent to the natural energy norm for the problem. The reader wishing to find complexity estimates for other norms should consult the monograph [13].

[2] Here we use the widely-used classification of finite element methods that was introduced by Babuška and his colleagues:

(1) h-methods, in which the degree of the finite element method is held fixed and the partition varies (these are the usual finite element methods),
(2) p-methods, in which the partition is fixed and the degree is allowed to vary,
(3) (h, p)-methods, in which the partition and degree are both allowed to vary.

See [1] for further discussion.

We now outline the contents of this paper. In Section 2, we precisely describe the problem to be solved. In Section 3, we find a lower bound of $\Omega\big(c(d)(\ln 1/\varepsilon)^d\big)$ on the problem complexity. In Section 4, we describe our finite element p-method and show that the cost of using this method to compute an ε-approximation is $O\big(c(d)(\ln 1/\varepsilon)^d\big)$. From these two results, we see that the problem complexity is $\Theta\big(c(d)(\ln 1/\varepsilon)^d\big)$ and that this finite element p-method is optimal, to within a constant factor that is independent of ε. Finally, in Section 5, we show that the problem is tractable in $1/\varepsilon$. Moreover, we briefly discuss issues relating to the tractability of the problem in d, i.e., the existence of $q > 0$ and a function $K(\cdot)$ such that $\mathrm{comp}(\varepsilon, d) \le c(d)K(\varepsilon)d^q$.

2. Problem description

In what follows, we assume that the reader is familiar with the usual terminology and notations arising in the variational study of elliptic boundary value problems. See Chapter 5 and the Appendix of [13] for further details, as well as the references cited therein.

Let $\mathscr{B}X$ denote the unit ball in the normed linear space X and let $\rho\mathscr{B}X$ denote the elements in X whose norm is at most ρ. In particular, we will let $\Omega = \mathscr{B}\mathbb{R}^d$ denote the real d-dimensional unit ball and $\Omega_\rho = \rho\mathscr{B}\mathbb{C}^d$ denote the d-dimensional complex ball of radius ρ, where $\rho > 1$. We use $\mathbb{N}$ to denote the nonnegative integers.

We now consider the partial differential operators defining our problem. Recall that for any $r \in \mathbb{R}$, the Sobolev space $H^r(\Omega)$ is the Hilbert space of all $L_2(\Omega)$-functions having all $L_2(\Omega)$-derivatives of order less than or equal to r (these spaces being defined by Hilbert space interpolation for nonintegral values of positive r and by duality for negative r). Also, recall that the space $H_0^r(\Omega)$ is the space of all $H^r(\Omega)$-functions having compact support. Let

$$Lv = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta} D^\beta v)$$

be a self-adjoint uniformly strongly elliptic partial differential operator on Ω_ρ, whose coefficient functions $a_{\alpha\beta} = a_{\beta\alpha}$ are analytic on Ω_ρ. We let $H^m_{\mathrm{bd}}(\Omega)$ be a closed subspace of $H^m(\Omega)$, containing $H^m_0(\Omega)$. That is, functions in $H^m_{\mathrm{bd}}(\Omega)$ satisfy a (possibly empty) set of homogeneous boundary conditions. Define a bilinear form B on $H^m_{\mathrm{bd}}(\Omega)$ as

$$B(v, w) = \sum_{|\alpha|,|\beta| \le m} \int_\Omega a_{\alpha\beta} D^\alpha v D^\beta w.$$

We finally assume that B is weakly coercive on $H^m_{\mathrm{bd}}(\Omega)$. This means that there exist positive constants γ_1 and γ_2 such that

$$\begin{aligned}
\inf_{\substack{v \in H^m_{\mathrm{bd}}(\Omega) \\ \|v\|_{H^m_{\mathrm{bd}}(\Omega)} \le 1}} \; \sup_{\substack{w \in H^m_{\mathrm{bd}}(\Omega) \\ \|w\|_{H^m_{\mathrm{bd}}(\Omega)} \le 1}} |B(v, w)| &\ge \gamma_1, \\
\sup_{\substack{v, w \in H^m_{\mathrm{bd}}(\Omega) \\ \|v\|_{H^m_{\mathrm{bd}}(\Omega)}, \|w\|_{H^m_{\mathrm{bd}}(\Omega)} \le 1}} |B(v, w)| &\le \gamma_2.
\end{aligned} \tag{2.1}$$

For example, suppose that $H^m_{\rm bd}(\Omega) = H^m_0(\Omega)$. Then B is weakly coercive on $H^m_0(\Omega)$ iff the only $v \in H^m_0(\Omega)$ for which $B(\cdot, v) = 0$ on $H^m_0(\Omega)$ is $v = 0$. See [13, Section 5.2] for further discussion.

Let

$$F = \{\, f \in \mathscr{B}H^\infty(\Omega_\rho) : x \in \Omega \implies f(x) \in \mathbb{R} \,\}.$$

We define a solution operator $S\colon F \to H^m_{\rm bd}(\Omega)$ by

$$B(Sf, v) = \langle f, v \rangle_{L_2(\Omega)}. \tag{2.2}$$

That is, $u = Sf$ is the variational solution to the problem

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega, \\ B_j u &= 0 \qquad \text{on } \partial\Omega \qquad (0 \le j \le m-1), \end{aligned} \tag{2.3}$$

where $(B_0, \ldots, B_{m-1})$ is a normal self-adjoint family of boundary operators, compatible with L. Since B is weakly coercive on $H^m_{\rm bd}(\Omega)$, the solution operator S is well-defined. Moreover, since all the data for our problem is analytic, the variational formulation (2.2) is equivalent to the classical formulation (2.3).

Remark: Note that Ω is the $\mathbb{R}^d$-region on which we measure error, while our class F of problem elements consists of functions analytic in the $\mathbb{C}^d$-region Ω_ρ. Note that we have chosen these regions as balls. This choice (which was made for expository purposes) is not as restrictive as it might seem. It is easy to see that the results of this paper also hold for more general regions satisfying the inclusion $\Omega \subset \Omega_\rho$ (which is necessary for the solution Sf to be defined for any $f \in F$), as well as a few mild geometric conditions. □

We assume that continuous linear information is permissible. Since our problem is linear (i.e., S is linear and F is convex and balanced), we may restrict ourselves to *nonadaptive* information of the form

$$Nf = [\lambda_1(f), \ldots, \lambda_n(f)] \qquad \forall f \in F, \tag{2.4}$$

where $\lambda_1, \ldots, \lambda_n$ are continuous linear functionals on F. Note that neither the number n of information evaluations, nor the choice of which functionals to evaluate will depend on the problem element f. (See [12, Chapter 4] for further discussion.)

Our model of computation is the standard one given in [12]. For any $f \in F$ and for any continuous linear functional λ, the cost of computing $\lambda(f)$ is $c(d)$. The cost of basic combinatory operations is 1. These basic operations include real arithmetic operations and comparisons, as well as the addition or scalar multiplication operations in $H^m_{\rm bd}(\Omega)$. Typically, $c(d) \gg 1$.

An *algorithm* using the information N is any map $\phi\colon F \to H^m_{\rm bd}(\Omega)$. Algorithms using N include, but are not limited to, the *linear algorithms* using N. For N defined by (2.4), these linear algorithms have the form

$$\phi^{\rm L}(Nf) = \sum_{j=1}^n D^{\alpha(j)} f(x_j) g_j \qquad \forall f \in F.$$

Here $g_1, \dots, g_n \in H^m_{\mathrm{bd}}(\Omega)$ depend only on the sample points $x_1, x_2, \dots, x_n$ and multi-indices $\alpha(1), \dots, \alpha(n)$ determining N, but are independent of any problem element $f \in F$. Note that once we have determined the functions $g_1, \dots, g_n$, we can evaluate $\phi^{\mathrm{L}}(Nf)$ with cost at most $\big(c(d)+2\big)n - 1$, for any f.

In this paper, we consider the worst case setting. Hence, the *error* of any algorithm ϕ using information N is given by

$$e(\phi, N) = \sup_{f \in F} \|Sf - \phi(Nf)\|_{H^m(\Omega)},$$

and the *cost* of this algorithm is given by

$$\mathrm{cost}(\phi, N) = \sup_{f \in F} \mathrm{cost}(\phi, N, f),$$

with $\mathrm{cost}(\phi, N, f)$ denoting the cost of computing ϕ for a particular problem element f. As always, the *ε-complexity*

$$\mathrm{comp}(\varepsilon, d) = \inf\{\, \mathrm{cost}(\phi, N) : e(\phi, N) \le \varepsilon \,\}$$

of our problem is the minimal cost of computing an ε-approximation, for $\varepsilon \ge 0$.

Let us recall a few standard results from the complexity theory of linear problems. Recall that for any information N, the *radius $r(N)$ of information* is the minimal error among all algorithms using N, i.e.,

$$r(N) = \inf_{\phi} e(\phi, N).$$

For any $n \in \mathbb{N}$, we let

$$r(n) = \inf\{\, r(N) : \mathrm{card}\, N \le n \,\}$$

denote the nth *minimal radius of information.* For any $\varepsilon \ge 0$, we let

$$m(\varepsilon) = \inf\{\, n \in \mathbb{N} : r(n) \le \varepsilon \,\}$$

denote the *ε-cardinality number.* Then

$$\mathrm{comp}(\varepsilon, d) \ge c(d)\, m(\varepsilon). \tag{2.5}$$

Moreover, suppose that for any information N, there exists a linear *optimal error algorithm* using N, i.e., a linear algorithm ϕ^{L} such that $e(\phi^L, N) = r(N)$. Then

$$\mathrm{comp}(\varepsilon, d) \le \big(c(d)+2\big)m(\varepsilon) - 1. \tag{2.6}$$

Hence (2.5) and (2.6) give a relation between the ε-cardinality number and the ε-complexity. For further details, see [12, Chapter 4].

3. The lower bound

In this section, we show that $\Omega\big(c(d)(\ln 1/\varepsilon)^d\big)$ is a lower bound on the problem complexity.

Before proving our lower bound, we need an auxiliary result on n-widths. Recall that if A is a convex, balanced subset of a normed linear space X and if $n \in \mathbb{N}$, then the *Gelfand n-width* of A in X is defined to be

$$d^n(A, X) = \inf_{L \in \mathscr{L}^n} \sup_{x \in A \cap L} \|x\|_X,$$

where $\mathscr{L}^n$ is the family of all subspaces of X whose codimension is at most n.

We also need to enumerate the multi-indices of $\mathbb{N}^d$ by increasing order. That is, we write the multi-indices as $\{\gamma(j)\}_{j=0}^{\infty}$, where

$$|\gamma(i)| \le |\gamma(j)| \qquad \text{if } i \le j.$$

Note that

$$\#\{\,\gamma \in \mathbb{N}^d : |\gamma| \le k\,\} = \dim \mathscr{P}_k(\mathbb{R}^d) = \binom{k+d}{d},$$

and so

$$|\gamma(n)| = \max\left\{\, k \in \mathbb{N} : \binom{k+d-1}{d} \le n \,\right\}. \tag{3.1}$$

Lemma 3.1. *Let $\eta = \rho e^2(2d^{1/2}+1)$ and $\kappa = 2^{d/2}d^{-d/4}(d+1)^{-d}$. Then*

$$d^n\big(\mathscr{B}H^\infty(\Omega_\rho), L_2(\Omega)\big) \ge \kappa\eta^{-|\gamma(n)|}.$$

Proof: Recall that the *Bernstein n-width* of A in X is defined to be

$$b_n(A, X) = \sup_{L \in \mathscr{L}_{n+1}} \inf_{x \in \partial(A \cap L)} \|x\|_X,$$

where $\mathscr{L}_{n+1}$ is the family of all subspaces of X whose dimension is at most $n+1$. Using [9, pg. 13] and [5, Proposition 2], we have

$$\begin{aligned} d^n\big(\mathscr{B}H^\infty(\Omega_\rho), L_2(\Omega)\big) \ge b_n\big(\mathscr{B}H^\infty(\Omega_\rho), L_2(\Omega)\big) &\ge \inf_{p \in \mathscr{P}_{|\gamma(n)|}} \frac{\|p\|_{L_2(\Omega)}}{\|p\|_{H^\infty(\Omega_\rho)}} \\ &\ge \rho^{-||\gamma(n)|} \inf_{p \in \mathscr{P}_{|\gamma(n)|}} \frac{\|p\|_{L_2(\Omega)}}{\|p\|_{H^\infty(\Omega_1)}}. \end{aligned} \tag{3.2}$$

Here $\mathscr{P}_{|\gamma(n)|}$ is the space of d-variable polynomials having real coefficents, whose degree is at most $|\gamma(n)|$, and Ω_1 is Ω_ρ for $\rho = 1$, i.e., the d-dimensional complex unit ball. Of course, $\mathscr{P}_{|\gamma(n)|}$ is a space of dimension at most $n+1$ over $\mathbb{R}$.

Let $p \in \mathscr{P}_{|\gamma(n)|}$. Then we may write

$$p(z) = \sum_{|\alpha| \le |\gamma(n)|} a_\alpha \prod_{j=1}^{d} P_{\alpha_j}(z_j d^{1/2}) \qquad \forall\, z \in \mathbb{C}^d,$$

where P_k is the kth Legendre polynomial. Using Laplace's first integral representation

$$P_k(\zeta) = \frac{1}{\pi}\int_0^\pi \left(\zeta + \cos\theta\sqrt{\zeta^2-1}\right)^k d\theta$$

(see [11, pg. 87]), we have

$$|P_k(\zeta)| \le \max_{0\le\theta\le 2\pi}\left|\zeta + \cos\theta\sqrt{\zeta^2-1}\right|^k \le \left(|\zeta| + \sqrt{|\zeta|^2-1}\right)^k \le (2|\zeta|+1)^k,$$

and so

$$\|p\|_{H^\infty(\Omega_1)} \le \sum_{|\alpha|\le|\gamma(n)|} |a_\alpha|(2d^{1/2}+1)^{|\alpha|} \le (2d^{1/2}+1)^{|\gamma(n)|}\sum_{|\alpha|\le|\gamma(n)|}|a_\alpha|. \tag{3.3}$$

Note that

$$n < \binom{|\gamma(n)|+d}{d} \le \frac{1}{d!}(|\gamma(n)|+d)^d \le \frac{1}{d!}|\gamma(n)|^d(d+1)^d \le e^{|\gamma(n)|}(d+1)^d. \tag{3.4}$$

Using the discrete Cauchy-Schwarz inequality, we see that

$$\sum_{|\alpha|\le|\gamma(n)|}|a_\alpha| \le (n+1)\left[\sum_{|\alpha|\le|\gamma(n)|}|a_\alpha|^2\right]^{1/2}. \tag{3.5}$$

Combining (3.3)–(3.5), we find that

$$\|p\|_{H^\infty(\Omega_1)} \le (2d^{1/2}+1)^{|\gamma(n)|}e^{|\gamma(n)|}(d+1)^d\left[\sum_{|\alpha|\le|\gamma(n)|}|a_\alpha|^2\right]^{1/2}. \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}
\|p\|^2_{L_2(\Omega)} &\ge \|p\|^2_{L_2([-d^{-1/2},d^{-1/2}]^d)} \\
&= \sum_{|\alpha|\le|\gamma(n)|} a_\alpha^2 \prod_{j=1}^d \int_{-d^{-1/2}}^{d^{-1/2}} P^2_{\alpha_j}(x_j d^{1/2})\,dx_j \\
&= \sum_{|\alpha|\le|\gamma(n)|} a_\alpha^2 \prod_{j=1}^d \left[d^{-1/2}\int_{-1}^1 P^2_{\alpha_j}(t)\,dt\right] \\
&= 2^d d^{-d/2}\sum_{|\alpha|\le|\gamma(n)|} a_\alpha^2 \prod_{j=1}^d \frac{1}{2\alpha_j+1}.
\end{aligned} \tag{3.7}$$

Using the arithmetic-geometric mean inequality, we have

$$\prod_{j=1}^d(2\alpha_j+1) \le \left[\frac{1}{d}\sum_{j=1}^d(2\alpha_j+1)\right]^d = \left(1+\frac{2|\alpha|}{d}\right)^d \le e^{2|\alpha|} \le e^{2|\gamma(n)|},$$

which, when combined with (3.7), yields

$$\|p\|_{L_2(\Omega)} \ge 2^{d/2}d^{-d/4}e^{-|\gamma(n)|}\left[\sum_{|\alpha|\le|\gamma(n)|}a_\alpha^2\right]^{1/2}.$$

Combining this inequality with (3.2) and (3.6), the desired result follows. □

We can now give a lower bound on the nth minimal radius for our problem:

THEOREM 3.1. *Let η be as in Lemma 3.1. There exists a constant $C_1 > 0$, independent of n, such that*

$$r(n) \geq C_1 \eta^{-|\gamma(n)|}$$

for all $n \in \mathbb{N}$.

PROOF: Since S is injective, we may use [12, Chapter 4, Theorem 5.4.1] to see that

$$r(n) = d^n\big(S(F), H^m_{\mathrm{bd}}(\Omega)\big), \tag{3.8.}$$

From (2.1), we find that

$$\|Sf\|_{H^m(\Omega)} \geq \gamma_1 \|f\|_{H^{-m}(\Omega)} \qquad \forall f \in H^{-m}(\Omega),$$

and so

$$d^n\big(S(F), H^m_{\mathrm{bd}}(\Omega)\big) \geq \gamma_1 d^n\big(F, H^{-m}(\Omega)\big). \tag{3.9}$$

To prove the lower bound, we let L^n be a subspace of $H^\infty(\Omega_\rho)$ whose codimension is at most n. Recall that the norms $\|\cdot\|_{H^m(\Omega)}$ and $\|(I-\Delta)^{m/2}\cdot\|_{L_2(\Omega)}$ are equivalent, see [2] or [8] for further discussion. Hence there exists a positive constant θ, depending only on m and d, such that

$$\|f\|_{H^{-m}(\Omega)} \geq \theta \|(I-\Delta)^{-m/2} f\|_{L_2(\Omega)}. \qquad \forall f \in H^{-m}(\Omega).$$

It now follows that

$$\sup_{f \in L^n \cap \mathscr{B}H^\infty(\Omega_\rho)} \|f\|_{H^{-m}(\Omega)} \geq \theta \sup_{g \in \tilde{L}^n \cap \mathscr{B}H^\infty(\Omega_\rho)} \|g\|_{L_2(\Omega)},$$

where $\tilde{L}^n = (I-\Delta)^{-m/2} L^n$ is a subspace of $H^\infty(\Omega_\rho)$ whose codimension is at most n. Since there is a bijection $L^n \leftrightarrow \tilde{L}^n$, we find

$$d^n\big(F, H^{-m}(\Omega)\big) \geq \theta\, d^n\big(\mathscr{B}H^\infty(\Omega_\rho), L_2(\Omega)\big) \geq \theta\kappa\eta^{-|\gamma(n)|},$$

the last being an application of Lemma 3.1. Using this result with (3.8) and (3.9), we find the desired lower bound, with $C_1 = \gamma_1\theta\kappa$. □

Using this lower bound on the nth minimal radius, we get the following estimate on the ε-complexity from (2.5) and (3.1):

COROLLARY 3.1. *Let C_1 and η be as in Theorem 3.1. Then*

$$\mathrm{comp}(\varepsilon, d) \geq c(d) \binom{\left\lceil \dfrac{\ln(C_1/\varepsilon)}{\ln \eta} \right\rceil + d - 1}{d} \geq \frac{c(d)}{d!} \left(\left\lceil \frac{\ln(C_1/\varepsilon)}{\ln \eta} \right\rceil - 1 \right)^d. \qquad \square$$

Hence we have shown a lower bound on the problem complexity, which is proportional to $c(d)\big(\ln(1/\varepsilon)\big)^d$, as promised.

4. Optimality of finite element methods

Since finite element methods (FEMs) have classically been among the most useful and widely-used algorithms for elliptic problems, we will seek an FEM that is a nearly optimal complexity algorithm. More precisely, we will show in this section how to construct a finite element p-method that can find an ε-approximation with cost proportional to $c(d)\big(\ln(1/\varepsilon)\big)^d$; from the bounds in the previous section, it follows that this p-FEM is nearly optimal.

Our FEM is described as follows.

Let $\mathscr{T}$ be a triangulation of Ω. Here, each $K \in \mathscr{T}$ is the affine image of a reference element $\hat{K}$ that is independent of K and $\mathscr{T}$. That is, there exists an affine bijection $F_K\colon \hat{K} \to K$.

For $k \in \mathbb{N}$, we let $\mathscr{P}_k(\hat{K})$ denote the space of all polynomials of degree at most k, considered as functions over $\hat{K}$. Letting $J = \dim \mathscr{P}_k(\hat{K})$, we choose points $\hat{x}_1, \dots, \hat{x}_J \in \hat{K}$; then there exist functions $\hat{p}_1, \dots, \hat{p}_J \in \mathscr{P}_k(\hat{K})$ such that $\hat{p}_i(\hat{x}_j) = \delta_{i,j}$ for $1 \le i, j \le J$. In what follows, we assume that there exists $M > 0$, independent of k, such that

$$\|\hat{p}_i\|_{L_\infty(\hat{K})} \le M. \tag{4.1}$$

(For example, this holds with $M = 1$ if we are using normalized tensor-product B-splines, see, e.g., [10, Chapters 4 and 12].)

Since each $K \in \mathscr{T}$ is the F_K-image of the reference element $\hat{K}$, we see that for any triangulation $\mathscr{T}$ and any element $K \in \mathscr{T}$, we can find a basis $\{p_{1,K}, \dots, p_{J,K}\}$ for $\mathscr{P}_k(K)$ by choosing $p_{i,K} = \hat{p}_i \circ F_K^{-1}$. Moreover, letting $x_{j,K} = F_K(\hat{x}_j)$ for $1 \le j \le J$, we find that $p_{i,K}(x_{j,K}) = \delta_{i,j}$ for $1 \le i, j \le J$.

Let

$$\mathscr{S}_{k,\mathscr{T}} = \left\{\, s \in H^m_{\text{bd}}(\Omega) : s\big|_K \in \mathscr{P}_k(K) \quad \forall\, K \in \mathscr{T} \,\right\}.$$

We say that $\mathscr{S}_{k,\mathscr{T}}$ is a *finite element space* of degree k over the triangulation $\mathscr{T}$.

Letting $n = \dim \mathscr{S}_{k,\mathscr{T}}$, we let $\{t_1, \dots, t_n\} = \{x_{1,K}, \dots, x_{J,K}\}_{K\in\mathscr{T}}$. We may choose a basis $\{s_1, \dots, s_n\}$ for $\mathscr{S}_{k,\mathscr{T}}$ by patching together the basis functions $\{p_{1,K}, \dots, p_{J,K}\}_{K\in\mathscr{T}}$, and then removing those basis functions that do not satisfy the boundary conditions of $H^m_{\text{bd}}(\Omega)$. (For more details, see [3, Chapter 2].) We then see that $s_j(t_i) = \delta_{i,j}$ for $1 \le i, j \le n$. Note that the *$\mathscr{S}_{k,\mathscr{T}}$-interpolation operator* $\Pi_{k,\mathscr{T}}$, defined as

$$\Pi_{k,\mathscr{T}} f = \sum_{j=1}^{n} f(t_j) s_j,$$

maps $C(\overline{\Omega})$ onto $\mathscr{S}_{k,\mathscr{T}}$.

For $f \in F$, we find $u_n \in \mathscr{S}_{k,\mathscr{T}}$ for which

$$B(u_n, s) = \langle \Pi_{k,\mathscr{T}} f, s\rangle_{L_2(\Omega)} \qquad \forall\, s \in \mathscr{S}_{k,\mathscr{T}}. \tag{4.2}$$

It is easy to check that u_n is well-defined, and that we can write

$$u_n = \phi_{n,k,\mathscr{T}}(N_{n,k,\mathscr{T}} f),$$

where

$$N_{n,k,\mathscr{T}} f = [f(t_1), \dots, f(t_n)].$$

The algorithm $\phi_{n,k,\mathscr{T}}$ is the *finite element method* (FEM) of degree k over $\mathscr{T}$, and $N_{n,k,\mathscr{T}}$ is the *finite element information* (FEI) that $\phi_{n,k,\mathscr{T}}$ uses.[3]

In what follows, we let $\{\mathscr{S}_{k,\mathscr{T}_n}\}_{n=1}^{\infty}$ be a family of finite element spaces of degree k, with $\dim \mathscr{S}_{k,\mathscr{T}_n} = n$. We assume that $\{\mathscr{T}_n\}_{n=1}^{\infty}$ is a *quasi-uniform* family of triangulations of Ω. This means that

$$\limsup_{n\to\infty} \sup_{K\in\mathscr{T}_n} \frac{h_K}{\rho_K} < \infty,$$

where

$$h_K = \operatorname{diam} K$$

and

$$\rho_K = \sup\{ \operatorname{diam} B : \text{spheres } B \text{ containing } K \}$$

for any region $K \subset \mathbb{R}^d$. In other words, we assume that the sequence of triangulations is not too irregular.

In what follows, we will respectively write $\phi_{n,k}$, $N_{n,k}$, $\mathscr{S}_{n,k}$, and $\Pi_{n,k}$ for $\phi_{n,k,\mathscr{T}_n}$, $N_{n,k,\mathscr{T}_n}$, $\mathscr{S}_{n,k,\mathscr{T}_n}$, and $\Pi_{n,k,\mathscr{T}_n}$.

We find an upper bound on the error of the FEM in

THEOREM 4.1. *There exists a positive constant C, independent of n, $\mathscr{T}_n$, k, and ρ, such that*

$$e(\phi_{n,k}, N_{n,k}) \le CMnk^{2m} \left(\frac{kn^{-1/d}}{\rho}\right)^{k+1}.$$

PROOF: Using Strang's Lemma ([13, Lemma A.3.2]), we have

$$\begin{aligned} \|Sf - u_n\|_{H^m(\Omega)} &\le C \left[\inf_{s\in\mathscr{S}_{k,\mathscr{T}_n}} \|Sf - s\|_{H^m(\Omega)} + \|f - \Pi_{n,k} f\|_{H^{-m}(\Omega)} \right] \\ &\le C \left[\|Sf - \Pi_{n,k} Sf\|_{H^m(\Omega)} + \|f - \Pi_{n,k} f\|_{L_2(\Omega)} \right] \end{aligned} \qquad (4.3).$$

Hence, we need only estimate the terms appearing on the right-hand side of (4.3).

Now

$$\|u - \Pi_{n,k} u\|_{H^m(\Omega)} \le \sqrt{m+1} \sum_{l=0}^{m} |u - \Pi_{n,k} u|_{H^l(\Omega)}$$

From [3, pg. 129], we have

$$|u - \Pi_{n,k} u|_{H^l(\Omega)} \le \hat{h}^m C_{k,l} \sum_{K\in\mathscr{T}_n} \frac{h_K^{k+1}}{\rho_K^m} |u|_{W^{k+1,\infty}(K)},$$

[3] Note that this, strictly speaking, not a "pure" FEM, but a "modified" FEM. In the "pure" FEM, the right-hand side of (4.2) would be $\langle f, s\rangle_{L_2(\Omega)}$ instead of $\langle \Pi_{k,\mathscr{T}} f, s\rangle_{L_2(\Omega)}$. That is, we would be computing inner products of the problem element f with the finite element basis functions. Since we wish to show that standard information consisting of function values is as strong as arbitrary continuous linear information, we need to modify the "pure" FEM to make use of standard information instead of inner products.

where $\hat{h} = \operatorname{diam} \hat{K}$ and

$$C_{k,l} = \frac{1}{(k+1)!} \sum_{i=1}^{J} |\hat{p}_i|_{W^{l,\infty}(\hat{K})}.$$

Since our family of triangulations is quasi-uniform, it follows that there exists a constant C, depending only on m and Ω, such that

$$\|u - \Pi_{n,k} u\|_{H^m(\Omega)} \le C \cdot \max_{0 \le l \le k} C_{k,l} h_n^{k+1-m} \, |\mathscr{T}_n| \, |u|_{W^{k+1,\infty}(\Omega)}, \tag{4.4}$$

where

$$h_n = \max_{K \in \mathscr{T}_n} h_K.$$

From [4], there exists a constant C such that for any $\hat{p} \in \mathscr{P}_k(\hat{K})$ and for any multi-index α with $|\alpha| = 1$, we have

$$\|D^\alpha \hat{p}\|_{L_\infty(\hat{K})} \le C k^2 d \|\hat{p}\|_{L_\infty(\hat{K})}.$$

It then follows that

$$|\hat{p}|_{W^{l,\infty}(\hat{K})} \le C k^{2l} d^l \|\hat{p}\|_{L_\infty(\hat{K})}$$

for a constant C independent of $\hat{p}$, K, and k. Since $J = \dim \mathscr{P}_k(\hat{K}) \le k^d$ and (4.1) holds, we thus find that

$$C_{k,l} \le \frac{CMk^{2m+d} d^m}{(k+1)!} \qquad (0 \le l \le m),$$

for a constant C, independent of k, l, n, and d.

Now $|\mathscr{T}_n| \cdot J = \Theta(n)$ and $J = \Theta(k^d)$. so that $|\mathscr{T}_n| = \Theta(nk^{-d})$. Hence

$$\begin{aligned} \|u - \Pi_{n,k} u\|_{H^m(\Omega)} &\le CMk^{2m+d} d^m n k^{-d} h_n^{k+1-m} \frac{|u|_{W^{k+1,\infty}(\Omega)}}{(k+1)!} \\ &\le CMk^{2m} n h_n^{k+1-m} \frac{|u|_{W^{k+1,\infty}(\Omega)}}{(k+1)!}, \end{aligned} \tag{4.5}$$

the constants C being independent of n, $\mathscr{T}_n$, k, and u. Since

$$\operatorname{vol}(\Omega) = \Theta(|\mathscr{T}_n| \cdot h_n^d),$$

we have

$$h_n = \Theta(|\mathscr{T}_n|^{-1/d}) = \Theta(kn^{-1/d}). \tag{4.6}$$

Using (4.4), (4.5), and (4.6), we thus have

$$\|u - \Pi_{n,k} u\|_{H^m(\Omega)} \le CMk^{2m} n (kn^{-1/d})^{k+1-m} \frac{|u|_{W^{k+1,\infty}(\Omega)}}{(k+1)!}, \tag{4.7}$$

the constant C being independent of n, k, and u.

From the analyticity results of [7], it follows that there are positive constants ρ and C, depending only on Ω and Ω_ρ, such that

$$\|D^\alpha u\|_{L_\infty(\Omega)} \le C\alpha!\rho^{-|\alpha|}\|u\|_{L_\infty(\Omega_\rho)} \le C\alpha!\rho^{-|\alpha|}\|f\|_{L_\infty(\Omega_\rho)},$$

for any multi-index α, and so

$$\frac{|u|_{W^{k+1,\infty}(\Omega)}}{(k+1)!} \le C\rho^{-(k+1)}.$$

Hence, (4.7) implies that

$$\|u - \Pi_{n,k}u\|_{H^m(\Omega)} \le CMk^{2m}n\left(\frac{kn^{-1/d}}{\rho}\right)^{k+1-m}.$$

Similarly, we find that

$$\|f - \Pi_{n,k}f\|_{L_2(\Omega)} \le CMn\left(\frac{kn^{-1/d}}{\rho}\right)^{k+1}.$$

The theorem now follows when we combine these last two inequalities with (4.3). □

We now show how to choose the degree k of the FEM $\phi_{n,k}$:

COROLLARY 4.1. *For $n \in \mathbb{N}$, let*

$$k(n) = \left\lceil \frac{\rho n^{1/d}}{e} \right\rceil,$$

with ρ as in Theorem 4.1. Let $\phi_n = \phi_{n,k(n)}$ and *$N_n = N_{n,k(n)}$. Then*

$$e(\phi_n, N_n) \le Cn^{2m/d+1}\sigma^{-n^{1/d}},$$

where

$$\sigma = \exp(\rho/e) \doteq 1.44467^\rho.$$

and the constant C is independent of n.

PROOF: Immediate from Theorem 4.1. □

We are now ready to find an upper bound on

$$\mathrm{cost}^{\mathrm{FE}}(\varepsilon, d) = \inf\{\, \mathrm{cost}(\phi_{n,k}, N_{n,k}) : e(\phi_{n,k}, N_{n,k}) \le \varepsilon \,\},$$

the minimal cost of using an FEM to compute an ε-approximation.

Corollary 4.2. *Let C and σ be as in Corollary 4.1. For $\varepsilon > 0$, let*

$$n = \left\lceil \left[\frac{\ln C/\varepsilon}{\ln \sigma} \left(1 + \frac{2m+d}{\ln C/\varepsilon - (2m+d)} \ln \left(\frac{\ln C/\varepsilon}{\ln \sigma} \right) \right) \right]^d \right\rceil . \tag{4.8}$$

Define ϕ_n and N_n as in Corollary 4.1. Then

$$e(\phi_n, N_n) \le \varepsilon$$

and

$$\mathrm{cost}(\phi_n, N_n) \le \big(c(d) + 2\big) \left\lceil \left[\frac{\ln C/\varepsilon}{\ln \sigma} \left(1 + \frac{2m+d}{\ln C/\varepsilon - (2m+d)} \ln \left(\frac{\ln C/\varepsilon}{\ln \sigma} \right) \right) \right]^d \right\rceil - 1,$$

so that

$$\mathrm{cost}^{\mathrm{FE}}(\varepsilon, d) = O\big(c(d)(\ln 1/\varepsilon)^d\big).$$

Proof: Clearly the bound on $\mathrm{cost}(\phi_n, N_n)$ follows from the choice of n. To complete the proof of the theorem, it suffices to show that $e(\phi_n, N_n) \le \varepsilon$.

Let

$$\begin{aligned} \xi &= n^{1/d}, \\ \alpha &= \frac{\ln C/\varepsilon}{\ln \sigma}, \\ \beta &= \frac{2m+d}{\ln \sigma}, \end{aligned} \tag{4.9}$$

so that

$$\xi = \alpha \left(1 + \frac{\beta}{\alpha - \beta} \ln \alpha \right).$$

Using the inequality

$$\ln(1+\delta) \le \delta \qquad \forall\, \delta \ge 0,$$

it easily follows that

$$\xi - \beta \ln \xi \ge \alpha.$$

Since this inequality holds, with α, β, and ξ given by (4.9), we may use Corollary 4.1 to find that

$$e(\phi_n, N_n) \le C n \sigma^{-n^{1/d}} \le \varepsilon,$$

which completes the proof of the theorem. □

Remark: We note that it is possible to choose a less complicated formula for n than that given by (4.8). Let $\sigma_1 < \sigma$, and choose

$$n = \left\lceil \left(\frac{\ln C/\varepsilon}{\ln \sigma_1} \right)^d \right\rceil . \tag{4.10}$$

Then it is easy to show that $\mathrm{cost}(\phi_n, N_n) = O\big(c(d)(\ln 1/\varepsilon)^d\big)$ and that $e(\phi_n, N_n) \le \varepsilon$ if $\varepsilon \le \varepsilon_0$, for some ε_0 depending only on σ_1, σ, and d. Moreover, if we choose n by the formula (4.10) instead of by the formula (4.8), the cost of the FEM ϕ_n will be less. The disadvantage of choosing n by (4.10) is that $e(\phi_n, N_n) \le \varepsilon$ will hold for a smaller range of ε than if we were to choose n by (4.8).

Using Corollaries 3.1 and 4.2, we immediately have the main result of this paper:

Corollary 4.3. *We have*

$$\mathrm{comp}(\varepsilon, d) = \Theta\big(c(d)(\ln 1/\varepsilon)^d\big)$$

and

$$\mathrm{cost}^{\mathrm{FE}}(\varepsilon, d) = \Theta\big(c(d)(\ln 1/\varepsilon)^d\big).$$

Moreover, the p-FEM of Corollary 4.2 is (to within a constant factor, independent of ε) an optimal complexity algorithm. □

5. Remarks on tractability

We are now ready to discuss tractability. Recall that a problem is *tractable* in $1/\varepsilon$ if there exists $p \geq 0$ and a function $K(\cdot)$ such that $\mathrm{comp}(\varepsilon, d) \leq c(d)K(d)(1/\varepsilon)^p$ for all d and ε. The infimum of such exponents p is said to be the *exponent* of the problem. For further details, see [16] and [17].

Corollary 5.1. *The problem is tractable in $1/\varepsilon$, with exponent 0.*

Proof: From Corollary 4.3, we have

$$\mathrm{comp}(\varepsilon, d) \leq c(d)\tilde{K}(d)\,(\ln 1/\varepsilon)^d\,,$$

for some function $\tilde{K}(\cdot)$. Note that for any $p > 0$, there exists $\varepsilon_0 = \varepsilon_0(p, d)$ such that the right-hand side of the previous inequality is bounded from above by $c(d)\tilde{K}(d)(1/\varepsilon)^p$ for $0 < \varepsilon < \varepsilon_0$. For $\varepsilon > 0$, we have

$$\mathrm{comp}(\varepsilon, d) \leq \max\left\{\mathrm{comp}(\varepsilon_0), \big(c(d) + 2\big)\tilde{K}(d)\left(\frac{1}{\varepsilon}\right)^p\right\} \leq K(d)c(d)\left(\frac{1}{\varepsilon}\right)^p,$$

with

$$K(d) = \frac{\max\{\varepsilon_0^p\,\mathrm{comp}(\varepsilon_0, d), K(d)\big(c(d) + 2\big)\}}{c(d)}.$$

Since $p > 0$ may be be chosen arbitrarily close to zero, the desired result follows. □

We now consider tractability in d. That is, we ask whether there exists $q \geq 0$ and a function $K(\cdot)$ such that $\mathrm{comp}(\varepsilon, d) \leq c(d)K(\varepsilon)d^q$ for all d and ε, the infimum of such q being called the *exponent* of the problem. Unfortunately, we cannot give a definitive answer to this question for our problem.

The reasons why are both technical and procedural. The technical reason is easy to explain. The lower and upper bounds respectively given by Corollaries 3.1 and 4.2 are close to each other only when d is fixed and ε decreases. If, on the other hand, we fix ε and let d increase, we find that the ratio of the upper bound to the lower bound increases (rapidly) with d, and so our bounds are no longer tight. Even worse, we find that our lower bound is too weak for proving d-intractability and our upper bound is too weak for proving d-tractability.

If this were the entire story, we could end this paper by saying that further work is needed on sharpening the bounds, i.e., that some technical matters need to be cleared up. However, there are some subtle procedural issues that are muddying the waters.

Note that we have a sequence of problems, each defined over a d-dimensional unit ball. Recall that the Lebesgue measure of the d-dimensional unit ball is $\pi^{d/2}/\Gamma(\frac{1}{2}d+1)$, which goes to zero rapidly with d. We are measuring the error of an approximation by an integral (with respect to this standard Lebesgue measure) over that domain. Suppose we are able to establish d-tractability. Is that tractability merely a reflection of the fact that the measures of the unit balls are shrinking?

One way around this problem is to not use integral norms as our measures of error. Instead, we can use sup-norms.

Example: Consider the problem of approximating functions, the error in an approximation being its maximum error over the *complex* d-dimensional unit ball. Formally speaking, we may specify this as a problem whose solution operator $S\colon F \to H^\infty(\Omega_1)$ is given by

$$Sf = f \qquad \forall f \in F,$$

where Ω_1 (which is Ω_ρ for $\rho = 1$) is the complex d-dimensional unit ball and (as before) $F = \mathscr{B}H^\infty(\Omega_\rho)$. Note that this problem is an elliptic problem of order $m = 0$.

For this problem, Farkov [5] proved that

$$d^n\bigl(F, L_\infty(\Omega_1)\bigr) = \rho^{-|\gamma(n)|},$$

and that there exist functions $h_0, \dots, h_{n-1} \in H^\infty(\Omega_\rho)$ such that

$$\sup_{f\in F} \left\| f - \sum_{j=0}^{n-1} D^{\gamma(j)} f(0) h_j \right\|_{L_\infty(\Omega_1)} = \rho^{-|\gamma(n)|}.$$

As a result, it follows that

$$c(d)\left(\begin{matrix} \left\lceil \dfrac{\ln(1/\varepsilon)}{\ln\rho} \right\rceil + d - 1 \\ d \end{matrix}\right) \le \mathrm{comp}(\varepsilon, d) \le \bigl(c(d)+2\bigr)\left(\begin{matrix} \left\lceil \dfrac{\ln(1/\varepsilon)}{\ln\rho} \right\rceil + d - 1 \\ d \end{matrix}\right) - 1.$$

We claim that this problem is intractable in d. Indeed, suppose that the problem is tractable in d, with exponent q. Choose $\varepsilon = \rho^{-(q+2)}$. Then

$$\mathrm{comp}(\varepsilon, d) \ge c(d)\binom{q+d+1}{d}$$

for all positive integers d. Since the binomial coefficient in the lower bound is a polynomial of degree $d+1$, we have a contradiction. Hence this complex approximation problem is intractable in d. $\square$

In view of all this, it seems that we need to pay serious attention to how the elliptic problem itself depends on the dimension d if we wish to seriously discuss tractability in d. In some sense, we want to be able to say that each of the d-dimensional problems is an instance of the "same" problem. Clearly, this topic requires further research.

Bibliography

1. Babuška, I., *Recent progress in the p and (h,p)-versions of the finite element method*, Numerical analysis 1987, (D. F. Griffiths and G. A. Watson, eds.), Wiley, New York, 1988, pp. 1–17.
2. Babuška, I. and Aziz, A. K., *Survey lectures on the mathematical foundations of the finite element method*, The mathematical foundations of the finite element method with applications to partial differential equations, (A. K. Aziz, ed.), Academic Press, New York, 1972, pp. 3–359.
3. Ciarlet, P. G., *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.
4. Coatmelec, C., *Approximation et interpolation des fonctions différentiables de plusieurs variables*, Ann. Sci. Ecole Norm. Sup. **83** (1966), 271–341.
5. Farkov, Yu. A., *Widths of Hardy classes and Bergman classes on the ball in* $\mathbb{C}^n$, Russian Mathematical Surveys **45** no. 5 (1990), 229–231.
6. Garey, M. R. and Johnson, D. S., *Computers and Intractability*, Freeman, San Francisco, 1979.
7. Morrey, C. B. and Nirenberg, L., *On the analyticity of solutions of linear elliptic systems of partial differential equations*, Communications on Pure and Applied Mathematics **10** (1957), 271–290.
8. Oden, J. T. and Reddy, J. N., *An introduction to the mathematical theory of finite elements*, Wiley-Interscience, New York, 1976.
9. Pinkus, A., *n-Widths in Approximation Theory*, Springer, Berlin, 1985.
10. Schumaker, L., *Spline Functions: Basic Theory*, Wiley, New York, 1980.
11. Szegö, G., *Orthogonal Polynomials*, AMS Colloquium Publications, Vol. 23, American Mathematical Society, Providence, RI, 1939.
12. Traub, J. F., Wasilkowski, G. W., and Woźniakowski, H., *Information-Based Complexity*, Academic Press, New York, 1988.
13. Werschulz, A. G., *The Computational Complexity of Differential and Integral Equations: An Information-Based Approach*, Oxford University Press, Oxford, 1991.
14. ———, *The complexity of two-point boundary-value problems with analytic data*, Journal of Complexity **9** (1993), 154–170.
15. ———, *The complexity of two-point boundary-value problems with piecewise analytic data*, submitted to Journal of Complexity (1993).
16. Woźniakowski, H., *Average case complexity of linear multivariate problems*; *I: Theory*, Journal of Complexity **8** (1992), 337–372; *II: Applications*, Journal of Complexity **8** (1992), 373–392.
17. ———, *Tracatability and strong tractability of linear multivariate problems*, Columbia University Computer Science Department Technical Report (1993).