Surface Approximation is Sometimes Easier Than Surface Integration

Arthur G. Werschulz*
Department of Computer and Information Sciences
Fordham University, New York, NY 10023
Department of Computer Science
Columbia University, New York, NY 10027

Henryk Woźniakowski†
Department of Computer Science
Columbia University, New York, NY 10027
Institute of Applied Mathematics
University of Warsaw, Poland

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Abstract

The approximation and integration problems consist of finding an approximation to a function f or its integral over some fixed domain Σ. For the classical version of these problems, we have partial information about the functions f and complete information about the domain Σ; for example, Σ might be a cube or ball in $\mathbb{R}^d$. When this holds, it is generally the case that integration is not harder than approximation; moreover, integration can be much easier than approximation. What happens if we have partial information about Σ? This paper studies the surface approximation and surface integration problems, in which $\Sigma = \Sigma_g$ for functions g. More specifically, the functions f are r times continuously differentiable scalar functions of

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l variables, and the functions g are s times continuously differentiable injective functions of d variables with l components. The class of surfaces considered is generated as images of cubes or balls, or as oriented cellulated regions. Error for the surface approximation problem is measured in the L_q-sense. These problems are well-defined, provided that $d \le l$, $r \ge 0$, and $s \ge 1$. Information consists of function evaluations of f and g. We show that the ε-complexity of surface approximation is proportional to $(1/\varepsilon)^{1/\mu}$ with $\mu = \min\{r, s\}/d$. We also show that if $s \ge 2$, then the ε-complexity of surface integration is proportional to $(1/\varepsilon)^{1/\nu}$ with

$$\nu = \min\left\{\frac{r}{d}, \frac{s - \delta_{s,1}(1 - \delta_{d,l})}{\min\{d, l-1\}}\right\}.$$

(This bound holds as well for several subcases of $s = 1$; we conjecture that it holds for all $r \ge 0$, $s \ge 1$, and $d \le l$.) Using these results, we determine when surface approximation is easier than, as easy as, or harder than, surface integration; all three possibilities can occur. In particular, we find that if $r = s = 1$ and $d < l$, then $\mu = 1/d$ and $\nu = 0$, so that surface integration is unsolvable and surface approximation is solvable; this is an extreme case for which surface approximation is easier than surface integration.

1 Introduction

Integration and approximation are two of the most well-studied problems of information-based complexity (IBC). These problems consist of finding an approximation to a function f or its integral over some domain Σ, typically using function values at a finite set of points from Σ. Usually,[1] one chooses Σ to be either the unit cube $I^d = [0, 1]^d$ or the Euclidean unit ball B_d in $\mathbb{R}^d$. For the usual sets F of problem elements f, these problems are *linear*. This means that we can use the full power of IBC theory for linear problems (see, e.g., [8, Sect. 4.5]), which allows us to determine $e(n)$ and $\mathrm{comp}(\varepsilon)$, the nth minimal error and the ε-complexity of approximation and integration, as well as to address issues such as whether linear algorithms are optimal and whether adaption is more powerful than non-adaption.

Usually, classical integration is no harder than approximation. Indeed, we can easily see this if we measure approximation error in a norm that dominates the $L_1(\Sigma)$-norm and if the Lebesgue measure of Σ is finite. Given any algorithm A for the approximation problem, consider the algorithm U defined by

$$U(f) = \int_\Sigma A(f)(x)\,dx.$$

[1] There is also a stream of work studying weighted approximation and integration over $\Sigma = \mathbb{R}^d$.

Then the error of U for the integration problem is dominated by the error of A for the approximation problem. It then follows that the nth minimal error of the integration problem is dominated by that of the approximation problem. In many cases, it is enough to only consider linear algorithms, so that the ε-complexity of integration is also dominated by that of approximation. Thus, integration is no harder than approximation.

In fact, integration may be easier than approximation. As an example, suppose that $\Sigma = I^d$, our functions to be approximated or integrated are a ball in the space $C^r(I^d)$ of r times continuously differentiable functions, and that we measure approximation error in the $C^{\tilde{r}}(I^d)$-norm, where $\tilde{r} < r$. Let the nth minimal error for a problem be defined as the minimal worst case error among all algorithms that use n function values to compute an approximation to that problem. Then the nth minimal errors for approximation and integration are $\Theta(n^{-(r-\tilde{r})/d})$ and $\Theta(n^{-r/d})$, respectively. Hence, integration is easier than approximation if $\tilde{r} > 0$.

Note that for these classical versions of the approximation and integration problems, we have *complete* information about the domain Σ. What happens if we only have *partial* information about Σ?

We consider nondegenerate domains Σ_g defined by functions g of d variables, nondegeneracy holding if g is a continuously differentiable injection. This necessarily implies that $d \le l$, where l is the dimension of the codomain of our functions g. Let G be a class of such functions g. The simplest situation is to let $\Sigma_g = g(I^d)$, where G is a class of functions from I^d to I^l, as was studied in [11] and [12]. Note that such a domain Σ_g is the diffeomorphic image of a cube, and hence Σ_g must have corners. If we want to allow smooth domains, we can follow one of two approaches. The first is to consider domains Σ_g that are oriented cellulated regions, see, e.g., [5, pp. 369–370]. This essentially means that Σ_g is a finite union of images of cubes under maps $g^{[1]}, \dots, g^{[k]} \in G$, with $g^{[1]}(I^d), \dots, g^{[k]}(I^d)$ having disjoint interiors. Examples of oriented cellulated regions include balls and spheres, as well as other smooth regions. It is easy to see that results that hold for domains that are images of cubes also hold for domains that are oriented cellulated regions. The difficulty is in actually constructing the necessary maps, even for regions as simple as balls. So, we propose a second approach, namely, letting $\Sigma_g = g(B_d)$ where G is a class of functions from a d-dimensional ball B_d to an l-dimensional ball B_l. Then the smoothness of $g(B_d)$ matches that of g.

Note that if $d < l$, then Σ_g is usually called a *surface* in $\mathbb{R}^l$, whereas if $d = l$, then Σ_g is a *region* in $\mathbb{R}^d$. For brevity's sake, we shall refer to Σ_g as a "surface" in all cases.

In this new setting, the *surface approximation problem* SURF-APP consists of approximating f over Σ_g in the L_q-sense, and the *surface integration problem*

SURF-INT consists of approximating the surface integral (as studied in [11]) of f over Σ_g. In both cases, we have only *partial* information about Σ_g since we can sample the function g only at finitely many points. We measure error and cost in the worst case setting, the worst case being over all $[f, g] \in F \times G$, so that F is a class of integrands and G is a class of functions g defining surfaces Σ_g.

In this paper, we shall choose F as a ball in the space of functions that are r times continuously differentiable. We shall choose G as a ball in a space of functions that are s times continuously differentiable injections, subject to a "uniform nondegeneracy" condition, explained in Section 2. We will require $r \geq 0$ and $s \geq 1$, so that our surface approximation and integration problems will be well-defined. We stress that both the surface approximation and surface integration problems are *nonlinear*.

Permissible information about $[f, g] \in F \times G$ will be standard information. For the purpose of simplifying the exposition, we restrict our attention to information consisting of function evaluations of f or g. There is no loss of generality in doing this, since the results of this paper also hold if we allow derivative evaluations. We let $\mathbf{c}$ denote the cost of one function evaluation.

Before explaining our results in a more technical way, we stress that surface approximation may be easier than, as easy as, or harder than surface integration, depending on the values of the global parameters. In particular, if the surface has minimal smoothness ($s = 1$) and $d < l$, then we have an extreme case. Namely, surface integration is unsolvable (i.e., its ε-complexity is infinite for small ε), but surface approximation is solvable (i.e., its ε-complexity is finite for all $\varepsilon > 0$). The intuitive reason for this is that a surface integral can be regarded as a weighted classical integral, whose weight depends on the derivative of the function g determining the surface Σ_g. When $s = 1$, this weight is only continuous. This is not enough smoothness to allow us to approximate the weight function with arbitrarily small error using a finite number of evaluations of g. Surprisingly enough, this holds only when $d < l$. However, when $d = l$, we can overcome this difficulty by using an integration by parts to reduce the dimension of the problem and to redefine the weight function so that it no longer contains derivatives of g. This new weight function is continuously differentiable, and hence it can be approximated with arbitrarily small error using finitely many function values of g. On the other hand, the same weight plays a completely different role for surface approximation. This weight only affects the error of an algorithm through a multiplicative factor. These factors are uniformly bounded for our class of surface approximation problems. From this, it follows that surface approximation always has finite ε-complexity, even when $s = 1$ and $d < l$.

Let us now state the main results of this paper more precisely. These results hold, no matter how our class of surfaces is generated:

- as images of cubes,
- as oriented cellulated regions, or
- as images of balls.

Note that we concentrate specifically on establishing estimates of the nth minimal error and ε-complexity having sharp exponents of $1/n$ and $1/\varepsilon$, ignoring any dependence of Θ-factors on r, s, d, and l. We believe that these factors depend exponentially on d and l, and so these estimates are of practical importance only for problems of small dimension.

First, we state the results for surface approximation. Let

$$\mu = \frac{\min\{r, s\}}{d}.$$

Then the nth minimal error satisfies[2]

$$e(n; \text{SURF-APP}) \asymp n^{-\mu}$$

and the ε-complexity satisfies[3]

$$\mathrm{comp}(\varepsilon; \text{SURF-APP}) \asymp \mathbf{c}\ \varepsilon^{-1/\mu}.$$

In particular, note that for classical approximation over balls and spheres, g is the identity function, and so we formally have $s = \infty$, so that $\mu = r/d$.

Next, we turn to surface integration. Let

$$\nu = \min\left\{\frac{r}{d}, \frac{s - \delta_{s,1}(1 - \delta_{d,l})}{\min\{d, l-1\}}\right\},$$

with $\delta_{i,j}$ is the usual Kronecker notation. Suppose that any of the following hold:

- $r = 0$,
- $d < l$,
- $d = l = 1$,
- $d = l \geq 2$ and $s \geq 2$,
- $d = l = 2$ and $s = 1$, or

[2]We use $\preccurlyeq$, $\succcurlyeq$, and $\asymp$ in this paper to respectively denote O-, Ω-, and Θ-relations.

[3]Here, we adopt the convention that $\varepsilon^{-1/0} = \infty$, so that $\mathrm{comp}(\varepsilon) \asymp \mathbf{c}\ \varepsilon^{-1/0}$ means that the problem is unsolvable for sufficiently small ε.

- $d = l = 3$ and $r = s = 1$.

Then

$$e(n; \text{SURF-INT}) \asymp n^{-\nu} \tag{1}$$

and

$$\text{comp}(\varepsilon; \text{SURF-INT}) \asymp \mathbf{c}\ \varepsilon^{-1/\nu}. \tag{2}$$

We do not have tight bounds for surface integration in the cases

- $d = l = 3$ and $r > s = 1$ and
- $d = l \geq 4$ and $r \geq s = 1$,

but we do know that the problem is solvable, i.e., it always has finite ε-complexity for any $\varepsilon > 0$, see [11]. We conjecture that the bounds (1)–(2) hold for all values of d, l, r, and s, subject to our conditions $r \geq 0$, $s \geq 1$, and $d \leq l$.

Our result for surface integration is succinct, but cryptic. Let us decipher it. First, note that if $r = 0$, then $\nu = 0$, i.e., the problem is unsolvable. Hence, we restrict our attention to the case $r \geq 1$.

1. Let $l = 1$, so that $d = 1$ necessarily. Then $\nu = r$.
2. Let $l \geq 2$ and $d < l$.
   (a) If $s = 1$, then $\nu = 0$.
   (b) If $s \geq 2$, then $\nu = \min\{r, s\}/d$.
3. Let $d = l \geq 2$.
   (a) Let $s \geq 2$. Then $\nu = \min\{r/d, s/(d-1)\}$.
   (b) Let $s = 1$.
      i. If $d = 2$, then $\nu = 1$.
      ii. If $d = 3$ and $r = 1$, then $\nu = \frac{1}{3}$.

Of course, in the remaining cases, we do not know the exact order of the ε-complexity.

Using these results, we can compare the ε-complexities of surface approximation and surface integration. We will use the usual precedence notation, so that, e.g., SURF-APP $\prec$ SURF-INT will mean that surface approximation is strictly easier (in terms of ε-complexity) than surface integration. First, suppose that $r = 0$. Then $\mu = \nu = 0$, and so SURF-APP $\asymp$ SURF-INT, with neither problem being solvable. Hence, we restrict our attention to the case $r \geq 1$. We find the following:

1. Let $l = 1$, so that $d = 1$. Then $\mu = \min\{r, s\}$ and $\nu = r$, so that SURF-INT $\preccurlyeq$ SURF-APP. In particular, if $r > s$, then SURF-INT $\prec$ SURF-APP, whereas if $r \le s$, then SURF-INT $\asymp$ SURF-APP.

2. Let $l \ge 2$ and $d < l$.

   (a) If $s = 1$, then $\mu = 1/d > 0$ and $\nu = 0$. That is, surface approximation is solvable, but surface integration is unsolvable. Hence SURF-APP $\prec$ SURF-INT. This case justifies the title of our paper.

   (b) If $s \ge 2$, then $\mu = \nu = \min\{r, s\}/d$, and so SURF-APP $\asymp$ SURF-INT.

3. Let $d = l \ge 2$.

   (a) If $s \ge 2$, then $\mu = \min\{r, s\}/d$ and $\nu = \min\{r/d, s/(d-1)\}$.

      i. If $r \le s$, then $\mu = \nu = r/d$, and so SURF-APP $\asymp$ SURF-INT.

      ii. If $r > s$, then $\mu = s/d$. There are two possibilities:

         A. If $r \le sd/(d-1)$, then $\nu = r/d > s/d = \mu$.

         B. If $r > sd(d-1)$, then $\nu = s/(d-1) > s/d = \mu$.

      Hence SURF-INT $\prec$ SURF-APP in either case.

   (b) If $s = 1$, and suppose that (1)–(2) holds. In particular, this includes the following cases:

      - $d = l = 2$, and
      - $d = l = 3$ and $r = 1$.

      Then $\mu = 1/d$ and $\nu = \min\{r/d, 1/(d-1)\}$.

      i. If $r/d \ge 1/(d-1)$, then $\nu = 1/(d-1) > 1/d = \mu$, and so SURF-INT $\prec$ SURF-APP.

      ii. Let $r/d < 1/(d-1)$. Then $r < d/(d-2) \le 2$, so that $r = 1$. Hence $\nu = 1/d = \mu$, and so SURF-APP $\asymp$ SURF-INT.

Thus we find that surface approximation is sometimes easier than, sometimes as easy as, and sometimes harder than, surface integration. It all depends on the relationship between r, s, d, and l.

We now outline the structure of this paper. In Section 2, we give a precise description of the problems to be solved. In Sections 3 and 4, we prove our results for the surface approximation and integration problems over images of cubes. Finally, in Section 5, we extend our results from images of cubes to smooth surfaces defined as oriented cellulated regions or as images of balls.

2 Problem description

Let d and l be given positive integers, with $d \le l$. Let $I = [0, 1]$ denote the unit interval. If $g: I^d \to \mathbb{R}^l$ is a C^1 injection, then $g(I^d)$ is a d-dimensional *surface* in $\mathbb{R}^l$.

Remark. Strictly speaking, the set $g(I^d)$ is a surface only when $d < l$; when $d = l$, the set $g(I^d)$ is a region of $\mathbb{R}^d$. However, for the sake of brevity, we shall let the word “surface” include all the cases $d \le l$. $\square$

In this paper, we deal with nondegenerate surfaces. The nondegeneracy of the surface means that the surface area element associated with the surface never vanishes. We now explain this more precisely; see [5, p. 334 ff.] for further discussion.

For any C^1 injection $g: I^d \to \mathbb{R}^l$, the *gradient* $\nabla g: I^d \to \mathbb{R}^{l\times d}$ is defined by

$$[(\nabla g)(x)]_{i,j} = \frac{\partial g_i}{\partial x_j}(x) \qquad \text{for } i \in \{1,\dots,l\},\ j \in \{1,\dots,d\}, \text{ and } x \in I^d,$$

where g_i is the ith component of g.

Define

$$\sigma_g(x) = \sqrt{\det A(x)} \qquad \forall\, x \in I^d,$$

where

$$A(x) = [(\nabla g)(x)]^T [(\nabla g)(x)] \qquad \forall\, x \in I^d,$$

i.e., $A(x) = [a_{i,j}(x)]_{i,j=1}^d$ is the $d \times d$ matrix having components

$$a_{i,j}(x) = \sum_{k=1}^{l} \frac{\partial g_k}{\partial x_i}(x)\frac{\partial g_k}{\partial x_j}(x)$$

for $i, j \in \{1,\dots,d\}$ and $x \in I^d$. We call $\sigma_g(x)$ the *surface area element* of $g(I^d)$ at x.

We say that the surface $g(I^d)$ is *nondegenerate* if $\sigma_g(x) \ne 0$ for all $x \in I^d$.

Remark. Note that when $d = l$, we have the simplification

$$\sigma_g(x) = |\det[(\nabla g)(x)]| \qquad \forall\, x \in I^d.$$

Strictly speaking, we should call $\sigma_g(x)$ the “volume element” rather than the “surface area element” in this case. Again, for brevity’s sake, we shall let “surface area element” cover all the cases $d \le l$. $\square$

Next, we define two classes F and G of functions, as in [11]. The class F will define functions to be either approximated or integrated over surfaces defined by the class G. We use the standard notations for multi-indices, Sobolev spaces, etc., found in, e.g., [2, p. 11].

For a positive C_1 and for $r \geq 0$, we first define $F = F_{l,r,C_1}$ as the ball of radius C_1 of the space $C^r(I^l)$, i.e.,

$$\|f\|_{C^r(I^l)} \leq C_1 \qquad \forall f \in F.$$

Here,

$$\|f\|_{C^r(I^l)} = \max_{|\alpha| \leq r} \|D^\alpha f\|,$$

with $\|\cdot\|$ denoting the max norm.

For positive C_2 and c_2 and for $s \geq 1$, we now define $G = G_{d,l,s,C_2,c_2}$ as the class of s times continuously differentiable functions $g \in C^s(I^d; I^l)$ that satisfy

$$\|g\|_{C^s(I^d;I^l)} \leq C_2 \qquad \text{and} \qquad \min_{x \in I^d} \sigma_g(x) \geq c_2. \tag{3}$$

Here,

$$\|g\|_{C^s(I^d;I^l)} = \max_{1 \leq i \leq l} \|g_i\|_{C^s(I^d)},$$

where, as before, $g_1, \dots, g_l$ denote the components of g. For simplicity, we assume that $c_2 < 1 \leq C_2$. Letting $\mathrm{id}\colon I^d \to I^l$ be the standard embedding

$$\mathrm{id}(x_1, \dots, x_d) = (x_1, \dots, x_d, 0, \dots, 0) \qquad \forall\, x = (x_1, \dots, x_d) \in I^d, \tag{4}$$

we see that $\mathrm{id} \in G$. Note that the condition (3) imposes a “uniform nondegeneracy condition” on the admissible surfaces $g(I^d)$.

Observe that the functions from F have the common domain I^l and that $g(I^d) \subseteq I^l$ for all $g \in G$. This is why the *compositions*

$$(f \circ g)(x) = f\big(g(x)\big) \qquad \forall\, x \in I^d$$

are well-defined for $[f, g] \in F \times G$.

We are now ready to formally define our two problems.

The first problem is SURF-APP, the *surface approximation problem*. For a given $[f, g] \in F \times G$ and $x \in I^d$, we wish to compute $A(t)$ for $t = g(x)$ such that $A(t)$ approximates $f(t)$. Let $q \in [1, \infty]$. The L_q-error of this approximation is given as

$$\|f - A\|_{L_q(g(I^d))}.$$

Here, for $q < \infty$, we have

$$\|f - A\|_{L_q(g(I^d))} = \left[\int_{g(I^d)} |f(t) - A(t)|^q \, d\sigma(t)\right]^{1/q},$$

where the integral is the *surface integral* defined as

$$\int_{g(I^d)} h\,d\sigma = \int_{I^d} (h \circ g)\,\sigma_g \equiv \int_{I^d} h\big(g(x)\big)\,\sigma_g(x)\,dx \qquad \forall\,[h,g] \in F \times G.$$

Hence we have

$$\|f - A\|_{L_q(g(I^d))} = \left[\int_{I^d} \big|f\big(g(x)\big) - A\big(g(x)\big)\big|^q\,\sigma_g(x)\,dx\right]^{1/q}$$

when $g < \infty$. For $q = \infty$, we have

$$\|f - A\|_{L_\infty(g(I^d))} = \sup_{t \in g(I^d)} |f(t) - A(t)| = \sup_{x \in I^d} \big|f\big(g(x)\big) - A\big(g(x)\big)\big|$$

(the latter since $c_2 \le \sigma_g(x) \le d!\,C_2^d$ for $x \in I^d$ and $g \in G$). Thus for any value of $q \in [1, \infty]$, we find that

$$\|f - A\|_{L_q(g(I^d))} = \|\sigma_g^{1/q}(f \circ g - A \circ g)\|_{L_q(I^d)}.$$

Let us now write $U(f,g)(x) = A\big(g(x)\big)$. Since g is injective, there is a one-to-one correspondence between U and A.

Our surface approximation problem consists of finding, for $[f,g] \in F \times G$, a function $U(f,g) \in L_q(I^d)$ such that

$$e(U; \text{SURF-APP}) := \sup_{[f,g] \in F \times G} \|\sigma_g^{1/q}[f \circ g - U(f,g)]\|_{L_q(I^d)}$$

is small. Equivalently, we want to approximate $f \circ g$ over I^d in the $\sigma_g^{1/q}$-weighted L_q-norm.

Our second problem is SURF-INT, the *surface integration problem*. For $[f,g] \in F \times G$, we seek $U(f,g) \in \mathbb{R}$ such that

$$e(U; \text{SURF-INT}) := \sup_{[f,g] \in F \times G} \left| \int_{g(I^d)} f\,d\sigma - U(f,g) \right|$$

is small.

Since $F \subset C(I^l)$ and $G \subset C^1(I^d; I^l)$, it follows that for any $[f,g] \in F \times G$, we have $\sigma_g^{1/q}(f \circ g) \in C(I^d) \subset L_q(I^d)$ and $(f \circ g)\sigma_g$ is integrable. Hence our surface approximation and integration problems are well-defined.

Note that we define the error of U for both problems in the worst case setting.

Remark. If $d = l$ and we choose $g = \mathrm{id}$, these problems reduce to the classical approximation and integration problems. Indeed, surface approximation with $g = \mathrm{id}$ is merely the $L_q(I^d)$-approximation problem for $C^r(I^d)$, and surface integration with $g = \mathrm{id}$ is the integration problem for $C^r(I^d)$. Hence these problems generalize the usual classical problems of approximation and integration. □

To solve either problem, we need to know some information about $f \in F$ and $g \in G$. We will consider *standard information* consisting of evaluations of f and g, which has the form

$$y = [y_1, \ldots, y_n] = N(f, g),$$

where, for $1 \le i \le n$, we have either

$$y_i = f(x^{(i)}) \text{ for some } x^{(i)} \in I^l$$

or

$$y_i = g_{j_i}(x^{(i)}) \text{ for some } x^{(i)} \in I^d \text{ and } j_i \in \{1, \ldots, l\}.$$

The information may be adaptive (i.e., the choice of each $x^{(i)}$ may depend on the previously computed information $y_1, \ldots, y_{i-1}$) or it may be nonadpative. Adaptive information is allowed to have varying cardinality, i.e., $n = n(f, g)$ is allowed to depend on f and g, or fixed cardinality. The *cardinality* of our information N is defined as

$$\operatorname{card} N = \sup_{[f,g] \in F \times G} n(f, g).$$

Note that since $F \subset C(I^l)$ and $G \subset C^1(I^d; I^l)$, standard information is well-defined over $F \times G$. For details and further discussion, see, e.g., [8, Chapter 2].

Remark. Note that the permissible information consists of function values of f and g. One could allow the evaluation of derivatives, as well. We restrict ourselves to function values alone, since this makes the exposition much simpler. However, it is easy to see that the results of this paper also hold if derivative evaluations are allowed. $\square$

Hence, our approximate solution has the form

$$U(f, g) = \phi\big(N(f, g)\big). \tag{5}$$

Here, we have $\phi\colon N(F, G) \to L_q(I^d)$ for the surface approximation problem, and $\phi\colon N(F, G) \to \mathbb{R}$ for the surface integration problem.

The cost of computing $U(f, g)$ is defined as $\operatorname{cost}(U, f, g)$, which is the weighted sum of the total number of information evaluations of f and g, as well as the number of arithmetic operations and comparisons needed to compute either

- $U(f, g)$ for surface integration, or
- $U(f, g)(x)$ at any $x \in I^d$ for surface approximation.

More precisely, we assume that the evaluation of a function at any point in its domain costs $\mathbf{c}$. The cost of each arithmetic operation is taken as 1. Then $\mathrm{cost}(U, f, g)$ for U of the form (5) is $\mathbf{c}\, n + \tilde{n}$, where $\tilde{n}$ is defined as either

- the total number of arithmetic operations and comparisons needed to compute $U(f, g)$ for surface integration, or
- the supremum (over all $x \in I^d$) of the total number of arithmetic operations and comparisons needed to compute $U(f, g)(x)$ for surface approximation.

Here $\mathbf{c} \geq 1$, and usually it is realistic to assume that $\mathbf{c} \gg 1$; see once more [8, Chapter 2] or [9, Chapter 2] for details. Then

$$\mathrm{cost}(U) = \sup_{[f,g] \in F \times G} \mathrm{cost}(U, f, g)$$

is the worst case cost of U.

We may judge the quality of an approximation U using information of given cardinality by comparing its error to the minimal error possible among all approximations using information of the same cardinality. For fixed n, the *nth minimal error*

$$e(n) = \inf\{\, e(U) : U \text{ of the form (5) with card } N \leq n \,\}$$

is the minimal worst case error among all approximations using any information of cardinality at most n. An approximation U_n for which

$$U_n = \phi_n \circ N_n \text{ with } \mathrm{card}\, N_n \leq n \qquad \text{and} \qquad e(U_n) \asymp e(n)$$

is said to be an nth (asymptotically) *minimal error algorithm*.

Clearly, $\{e(n)\}$ is a nonincreasing sequence. Moreover, $e(n)$ makes sense even when $n = 0$; indeed, $e(0)$ is minimal error among all "constant" approximations, i.e., those using no evaluations of f and g.

Along with minimal-error approximations using a given number n of information evaluations, we also wish to compute ε-approximations at minimal cost for any $\varepsilon \geq 0$. The *ε-complexity* is the minimal cost of computing an ε-approximation, i.e.,

$$\mathrm{comp}(\varepsilon) = \inf\{\, \mathrm{cost}\, U \;:\; U \text{ such that } e(U) \leq \varepsilon \,\}.$$

An approximation U_ε for which

$$e(U_\varepsilon) \leq \varepsilon \qquad \text{and} \qquad \mathrm{cost}\, U_\varepsilon \asymp \mathrm{comp}(\varepsilon) \quad \text{as} \quad \varepsilon \to 0,$$

is said to be (asymptotically) *optimal*.

Remark. Of course, the error $e(U)$, the nth minimal error $e(n)$, and the ε-complexity also depend on which problem is being solved. We shall often denote this dependence explicitly, writing, e.g., $e(n; \text{SURF-APP})$ or $\mathrm{comp}(\varepsilon; \text{SURF-INT})$. On the other hand, even though these quantities also depend on the classes F and G, we shall not indicate these dependencies explicitly. $\square$

3 The surface approximation problem

In this section, we determine the nth minimal error and the ε-complexity of the surface approximation problem. Note that since the weight $\sigma_g^{1/q}$ is uniformly bounded from above and below for $g \in G$, this weight does not affect the order of the error of an approximation U. Hence the surface approximation problem will have the same complexity as the approximation problem for composite functions.

We first establish an upper bound for our problem, by exhibiting an algorithm using information of cardinality n whose error is proportional to $n^{-\mu}$, where

$$\mu = \frac{\min\{r, s\}}{d}.$$

Let us determine the smoothness of functions $f \circ g$ for $[f, g] \in F \times G$.

Lemma 3.1. *If $[f, g] \in F \times G$, then $f \circ g \in C^{\min\{r,s\}}(I^d)$. Moreover, there exists a constant $C_\circ$, independent of $[f, g] \in F \times G$, such that*

$$\|f \circ g\|_{C^{\min\{r,s\}}(I^d)} \le C_\circ.$$

Proof. Let $[f, g] \in F \times G$, and let α be a nonzero multi-index for which $|\alpha| \le \min\{r, s\}$. The multivariate Faa di Bruno formula [3, Theorem 2.1] states that

$$D^\alpha (f \circ g)(x) = \sum_{1 \le |\beta| \le |\alpha|} (D^\beta f)\big(g(x)\big) \sum_{i=1}^{|\alpha|} \sum_{p_i(\alpha,\beta)} \alpha! \prod_{j=1}^{i} \frac{[D^{\ell_j} g(x)]^{k_j}}{k_j!(\ell_j!)^{|k_j|}},$$

where $p_i(\alpha, \beta)$ is the set of nonzero multi-indices $k_1, \dots, k_i \in \mathbb{Z}^l$ and $\ell_1, \dots, \ell_i \in \mathbb{Z}^d$ (with the ℓ_j being strictly increasing with respect to lexicographic ordering) such that

$$\sum_{j=1}^{i} k_j = \beta \in \mathbb{Z}^l \qquad \text{and} \qquad \sum_{j=1}^{i} |k_j| \ell_j = \alpha \in \mathbb{Z}^d.$$

Hence we find that there exists a constant $C_{d,l,|\alpha|}$, independent of f, g, and n, such that

$$\begin{aligned}
\|D^\alpha(f\circ g)\|_{C(I^d)} &\le C_{d,l,|\alpha|}\sum_{1\le|\beta|\le|\alpha|}\|f\|_{C^{|\beta|}(I^d)}\|g\|^{|\beta|}_{C^{|\alpha|}(I^d)}\\
&\le C_{d,l,|\alpha|}\sum_{1\le|\beta|\le|\alpha|}C_1C_2^{|\beta|}\preccurlyeq 1,
\end{aligned}$$

where C_1 and C_2 appear in the definition of the classes F and G. The desired result follows immediately. $\square$

We now recall some standard results on approximation in the $L_q(I^d)$-norm. From [2, Chap. 3.1], there is an interpolation operator Π_n, having the form

$$\Pi_n w=\sum_{j=1}^n w(x^{(i)})s_i \qquad \forall\, w\in C^{\min\{r,s\}}(I^d),$$

with the following properties:

1. For any $x\in I^d$ and $i\in\{1,\dots,n\}$, the value $s_i(x)$ can be computed in $O(1)$ arithmetic operations.
2. There exists a number C_{APP}, depending only on the global parameters d, q, r, and s, such that

$$\|w-\Pi_n w\|_{L_q(I^d)}\le C_{\text{APP}}\,n^{-\mu}\,\|w\|_{C^{\min\{r,s\}}(I^d)} \tag{6}$$

   for any $w\in C^{\min\{r,s\}}(I^d)$ and any $n\ge 1$.

We now define an approximation U_n for our problem SURF-APP by taking

$$U_n(f,g)=\Pi_{\lfloor n/(l+1)\rfloor}(f\circ g)\qquad \forall\,[f,g]\in F\times G. \tag{7}$$

Clearly U_n uses the information

$$N_n(f,g)=[(f\circ g)(x^{(1)}),\dots,(f\circ g)(x^{\lfloor n/(l+1)\rfloor})]$$

of cardinality at most n.

Lemma 3.2. *For any $n\ge 0$, we have*

$$\operatorname{cost}(U_n)\preccurlyeq \boldsymbol{c}\ n$$

and

$$e(U_n;\text{SURF-APP})\preccurlyeq n^{-\mu}.$$

Proof. The bound on the cost of U_n follows immediately from the definition of U_n, the linearity of $\Pi_{\lfloor n/(l+1)\rfloor}$, and the form of the basis functions $s_1, \dots, s_n$. Now let $[f, g] \in F \times G$. Since $\sigma_g(x) \le d!C_2^d$, we may use (6) and Lemma 3.1 to see that

$$\begin{aligned}
\|\sigma_g^{1/q}[f \circ g - \Pi_{\lfloor n/(l+1)\rfloor}(f \circ g)]\|_{L_q(I^d)} \\
&\le (d!C_2^d)^{1/q}\, C_{\mathrm{APP}} \left\lfloor \frac{l+1}{n} \right\rfloor^{-\mu} \|f \circ g\|_{C^{\min\{r,s\}}(I^d)} \\
&\le (d!C_2^d)^{1/q}\, C_{\mathrm{APP}}\, C_\circ \left\lfloor \frac{l+1}{n} \right\rfloor^{-\mu}.
\end{aligned}$$

Hence $e(U_n) \preccurlyeq n^{-\mu}$, as required. ☐

We now establish lower bounds for our problem. The first is fairly simple:

Lemma 3.3.

$$e(n; \text{SURF-APP}) \succcurlyeq n^{-r/d}.$$

Proof. For any $v \in F_{d,r,C_1}$, we define $f_v \colon I^l \to \mathbb{R}$ as

$$f_v(x_1, \dots, x_d, x_{d+1}, \dots, x_l) = v(x_1, \dots, x_d) \qquad \forall\, x = (x_1, \dots, x_l) \in I^l.$$

Let $g = \mathrm{id} \in G$ be given by (4). Then $f_v \circ g = f_v \circ \mathrm{id} = v$ and

$$\|f_v\|_{C^r(I^l)} = \|v\|_{C^r(I^d)}.$$

Hence $f_v \in F = F_{l,r,C_1}$. For an approximation U of the surface approximation problem using information of cardinality at most n, we define an approximation $\tilde{U}$ to the classical approximation problem APP of approximating F_{d,r,C_1} in the $L_q(I^d)$ norm, by taking

$$\tilde{U}(v) = U(f_v, \mathrm{id}) \qquad \forall\, v \in F_{d,r,C_1}.$$

Clearly $\tilde{U}(v)$ uses information of cardinality at most n about v, and so we have

$$\begin{aligned}
e(U, \text{SURF-APP}) &\ge \sup_{f \in F} \|f \circ \mathrm{id} - U(f, \mathrm{id})\|_{L_q(I^d)} \\
&\ge \sup_{v \in F_{d,r,C_1}} \|f_v \circ \mathrm{id} - U(f_v, \mathrm{id})\|_{L_q(I^d)} \\
&= \sup_{v \in F_{d,r,C_1}} \|v - \tilde{U}(v)\|_{L_q(I^d)} \\
&= e(\tilde{U}, \text{APP}) \ge e(n; \text{APP}).
\end{aligned}$$

It is well-known, see, e.g., [6], we have

$$e(n; \text{APP}) \succcurlyeq n^{-r/d}.$$

The desired result follows from the previous two inequalities. ☐

Our second lower bound is somewhat more complicated to establish, since it is based on the class G, which is defined not only by the smoothness of the functions g, but also by restriction to g satisfying $g(I^d) \subseteq I^l$ and (3).

Lemma 3.4.

$$e(n; \text{SURF-APP}) \succcurlyeq n^{-s/d}.$$

Proof. Without loss of generality, let us assume that $C_1 \geq 1$. We first define the function $f^* \colon I^l \to \mathbb{R}$ as

$$f^*(x) = x_1 \qquad \forall\, x \in I^l.$$

Clearly $f^* \in F$.

Let

$$\xi = \frac{2d - 1 + 2c_2}{1 + 2d} \qquad \text{and} \qquad \eta = \frac{1 - c_2}{1 + 2d}.$$

Since $d \geq 1$ and $0 \leq c_2 < 1$, we have $\xi > 0$, $\eta > 0$, and $\xi + 2\eta = 1$. Using this and Taylor's theorem with remainder for the function $\eta \mapsto (1 - 2\eta)^{d-1}$ with $d \geq 2$, we have

$$\begin{aligned}(\xi - \eta)\xi^{d-1} &= (1 - 3\eta)(1 - 2\eta)^{d-1} \geq (1 - 3\eta)\big(1 - 2(d-1)\eta\big) \\ &\geq 1 - \big(3 + 2(d-1)\big)\eta = c_2.\end{aligned} \tag{8}$$

Let $e = (1, \ldots, 1)^T \in \mathbb{R}^l$. We define

$$g^* = \xi \,\mathrm{id} + \eta e.$$

We claim that $g^* \in G$.

1. We need to show that $g^* \colon I^d \to I^l$. Let $x \in I^d$. For $i \in \{1, \ldots, d\}$, we have $g_i^*(x) = \xi x_i + \eta$. Since ξ and η are positive and $\xi + \eta \leq 1$, we have $0 \leq g_i^*(x) \leq 1$. For $i \in \{d+1, \ldots, l\}$, we have $g_i^*(x) = \eta \in [0, 1]$. Hence $g^*(I^d) \subseteq I^l$, as required.

2. We need to show that $\|g^*\|_{C^s(I^d;I^l)} \leq C_2$. Since $C_2 \geq 1$, it is enough to prove that $\|g\|_{C^s(I^d;I^l)} \leq 1$. Now we have $\|g^*\|_{C(I^d;I^l)} = \xi + \eta$. If α is a multi-index with $|\alpha| = 1$, then $\|D^\alpha g^*\|_{C(I^d;I^l)} \leq \xi$, whereas if α is a multi-index with $|\alpha| \geq 2$, then $D^\alpha g^* = 0$. Hence $\|g^*\|_{C^s(I^d;I^l)} = \xi + \eta \leq 1$, as required.

3. We need to verify that $\min_{x \in I^d} \sigma_g(x) \geq c_2$. Now

$$\nabla g^* = \xi I_{l\times d} = \xi \begin{bmatrix} I_{d\times d} \\ 0_{(l-d)\times d} \end{bmatrix},$$

and so

$$A_{g^*} = (\nabla g^*)^T (\nabla g^*) = \xi^2 I_{d\times d}.$$

From (8), we have

$$\sigma_g = \sqrt{|\det A_g|} = \xi^d \ge (\xi - \eta)\xi^{d-1} \ge c_2,$$

as required.

Thus, we see that $g^* \in G$, as claimed.

Now let N be information of cardinality at most n. Without loss of generality, there exist non-negative integers $n_0, \dots, n_l$ with $\sum_{i=0}^{l} n_i \le n$, a set $\{x^{(0,j)}\}_{1\le j\le n_0}$ of points in I^l, and a set $\{x^{(i,j)}\}_{1\le j\le n_i, 1\le i\le l}$ of points in I^d, such that

$$N(f^*, g^*) = [N_0(f^*), N_1(g_1^*), \dots, N_d(g_d^*)],$$

where

$$N_0(f^*) = [f^*(x^{(0,1)}), \dots, f^*(x^{(0,n_0)})]$$

and

$$N_i(g_i^*) = [g_i^*(x^{(i,1)}), \dots, g_i^*(x^{(i,n_i)})] \qquad \text{for } 1 \le i \le l.$$

From [6], there exists a function $z \in C^s(I^d)$ such that

$$\begin{aligned} \|z\|_{C^s(I^d)} &= 1, \\ N_1(z) &= 0, \\ \|z\|_{L_q(I^d)} &\succcurlyeq n^{-s/d}. \end{aligned} \tag{9}$$

Let $e_1 \in \mathbb{R}^l$ denote the first column of the $l \times l$ identity matrix $I_{l\times l}$. We will take

$$g^{**} = g^* + \eta z e_1 = \xi\,\mathrm{id} + \eta e + \eta z e_1.$$

We claim that $g^{**} \in G$.

1. We need to show that $g^{**} \colon I^d \to I^l$. We have

$$g_1^{**}(x) = \xi x_1 + \eta + \eta z(x) \ge \eta(1 - |z(x)|) \ge 0$$

and

$$g_1^{**}(x) \le \xi + \eta + \eta |z(x)| \le \xi + 2\eta = 1.$$

Since $g_i^{**} = g_i^* \colon I^d \to I$ for $i \ge 2$, we have $g^{**} \colon I^d \to I^l$, as desired.

2. We need to show that $\|g^{**}\|_{C^s(I^d;I^l)} \leq C_2$. We have already showed that $\|g_1^{**}\|_{C(I^d)} \leq 1$. If α is a multi-index with $|\alpha| = 1$, then $\|D^\alpha g_1^{**}\|_{C(I^d)} \leq \xi + 2\eta$, whereas if α is a multi-index with $|\alpha| \in [2, s]$, then $\|D^\alpha g_1^{**}\|_{C(I^d)} \leq \eta$. On the other hand, if $2 \leq i \leq d$, we have $g_i^{**} = g_i^*$, and hence

$$\|D^\alpha g_i^{**}\|_{C(I^d)} \leq \begin{cases} \xi + \eta & \text{if } |\alpha| = 0, \\ \xi & \text{if } |\alpha| = 1, \\ 0 & \text{if } |\alpha| \geq 2. \end{cases}$$

Thus

$$\|g^{**}\|_{C^s(I^d;I^l)} \leq \xi + 2\eta = 1 \leq C_2,$$

as required.

3. We need $\min_{x \in I^d} \sigma_{g^{**}}(x) \geq c_2$. Now

$$\nabla g^{**} = \nabla g^* + \eta \nabla(ze_1) = \nabla g^* + \eta e_1 (\nabla z)^T = \xi I_{l\times d} + \eta \begin{bmatrix} (\nabla z)^T \\ 0_{(l-1)\times d} \end{bmatrix}.$$

Hence

$$\begin{aligned} A_{g^{**}} &= (\nabla g^{**})^T (\nabla g^{**}) \\ &= \left(\xi I_{d\times l} + \eta \begin{bmatrix} \nabla z & 0_{d\times(l-1)} \end{bmatrix}\right) \left(\xi I_{l\times d} + \eta \begin{bmatrix} (\nabla z)^T \\ 0_{(l-1)\times d} \end{bmatrix}\right) \\ &= \xi^2 I_{d\times d} + \xi\eta \left(\begin{bmatrix} \nabla z & 0_{d\times(d-1)} \end{bmatrix} + \begin{bmatrix} (\nabla z)^T \\ 0_{(d-1)\times d} \end{bmatrix} \right) + \eta^2 (\nabla z)(\nabla z)^T \\ &= [\tilde{a}_{i,j}]_{1\leq i,j\leq d}. \end{aligned}$$

Here,

$$\begin{aligned} \tilde{a}_{1,1} &= (\xi + z_{,1}\eta)^2 && \\ \tilde{a}_{i,i} &= \xi^2 + \eta^2 z_{,i}^2 && \text{for } 2 \leq i \leq d \\ \tilde{a}_{j,1} &= \tilde{a}_{1,j} = \eta z_{,j}(\xi + z_{,1}\eta) && \text{for } 2 \leq j \leq d \\ \tilde{a}_{j,i} &= \tilde{a}_{i,j} = \eta^2 z_{,i} z_{,j} && \text{for } 2 \leq i \neq j \leq d, \end{aligned}$$

with $z_{,j}$ being used to denote $\partial z/\partial x_j$. Since the first row and first column of $A_{g^{**}}$ are multiples of $\xi + z_{,1}\eta$, we have

$$\det A_{g^{**}} = (\xi + z_{,1}\eta)^2 \det B,$$

where

$$B = \begin{bmatrix} 1 & \eta z_{,2} & \eta z_{,3} & \dots & \eta z_{,d} \\ \eta z_{,2} & \xi^2 + \eta^2 z_{,2}^2 & \eta^2 z_{,2} z_{,3} & \dots & \eta^2 z_{,2} z_{,d} \\ \eta z_{,3} & \eta^2 z_{,2} z_{,3} & \xi^2 + \eta^2 z_{,3}^2 & \dots & \eta^2 z_{,3} z_{,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta z_{,d} & \eta^2 z_{,2} z_{,d} & z^2 z_{,3} z_{,d} & \dots & \xi^2 + \eta^2 z_{,d}^2 \end{bmatrix}.$$

Now for $2 \le i \le d$, subtract $\eta z_{,i}$ times the first row of B from the ith row of B. We get

$$\det B = \begin{vmatrix} 1 & \eta z_{,2} & \eta z_{,3} & \dots & \eta z_{,d} \\ 0 & \xi^2 & 0 & \dots & 0 \\ 0 & 0 & \xi^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \xi^2 \end{vmatrix} = \xi^{2(d-1)}.$$

Hence

$$\det A_{g^{**}} = (\xi + z_{,1}\eta)^2 \xi^{2(d-1)}.$$

Since $\|z\|_{C^s(I^d)} = 1$, we have

$$\sigma_{g^{**}}(x) = |\xi + z_{,1}(x)\eta|\xi^{d-1} \ge \big(\xi - \eta|z_{,1}(x)|\big)\,\xi^{d-1} \ge (\xi - \eta)\xi^{d-1} \ge c_2,$$

the latter by (8).

Thus, we see that $g^{**} \in G$, as claimed.

Using (9), we have $N_1(z) = 0$, which implies that

$$N(f^*, g^{**}) = N(f^*, g^*).$$

We also have

$$f^* \circ g^* - f^* \circ g^{**} = g_1^{**} - g_1^* = \eta z.$$

Now let ϕ be an algorithm using N. Then $w := \phi\big(N(f^*, g^{**})\big) = \phi\big(N(f^*, g^*)\big)$, and thus

$$\begin{aligned} e(\phi, N) &\ge \max\left\{\|\sigma_{g^*}^{1/q}(f^* \circ g^* - w)\|_{L_q(I^d)}, \|\sigma_{g^{**}}^{1/q}(f^* \circ g^{**} - w)\|_{L_q(I^d)}\right\} \\ &\ge c_2^{1/q} \max\left\{\|f^* \circ g^* - w\|_{L_q(I^d)}, \|f^* \circ g^{**} - w\|_{L_q(I^d)}\right\} \\ &\ge \tfrac{1}{2} c_2^{1/q}\left(\|f^* \circ g^* - w\|_{L_q(I^d)} + \|f^* \circ g^{**} - w\|_{L_q(I^d)}\right) \\ &\ge \tfrac{1}{2} c_2^{1/q} \|f^* \circ g^* - f^* \circ g^{**}\|_{L_q(I^d)} = \tfrac{1}{2} c_2^{1/q} \eta \|z\|_{L_q(I^d)}. \end{aligned}$$

By (9), we conclude that

$$e(\phi, N) \succcurlyeq n^{-s/d}.$$

Since ϕ is an arbitrary algorithm using arbitrary information N of cardinality at most n, we see that

$$e(n; \text{SURF-APP}) \succcurlyeq n^{-s/d},$$

as required. $\square$

Combining Lemmas 3.2–3.4, we have

Theorem 3.1. *Consider the surface approximation problem with $F = F_{l,r,C_1}$ and $G = G_{d,l,s,C_2,c_2}$. Let*

$$\mu = \frac{\min\{r, s\}}{d}.$$

1. *The nth minimal error satisfies*

$$e(n; \text{SURF-APP}) \asymp n^{-\mu}$$

and is attained by U_n defined by (7).

2. *The ε-complexity satisfies*

$$\mathrm{comp}(\varepsilon; F, G) \asymp \boldsymbol{c}\ \varepsilon^{-1/\mu}.$$

Moreover the approximation U_n, with $n \asymp \varepsilon^{-1/\mu}$, is optimal and computes an ε-approximation at nearly-minimal cost. $\square$

4 The surface integration problem

In this section, we determine the nth minimal error and the ε-complexity of the surface integration problem.

First, it is clear that for $r = 0$, the surface integration problem is unsolvable. That is,

$$e(n; \text{SURF-INT}) \asymp 1$$

and there exists $\varepsilon_0 > 0$ such that

$$\mathrm{comp}(\varepsilon, \text{SURF-INT}) = \infty \qquad \forall\, \varepsilon \leq \varepsilon_0.$$

Indeed, it is enough to take $g = \mathrm{id}$, so that surface integration reduces to the problem of approximating $\int_{I^d} h(x)\, dx$, where

$$h(x) = f(x, 0, \ldots, 0) \qquad \forall\, x \in I^d.$$

This problem is the classical integration problem for continuous functions over I^d; it is well-known (see, e.g., [1]) that this problem is unsolvable. Therefore, we will restrict our attention to the case $r \geq 1$ in Theorems 4.1 and 4.2.

Our main result is the following theorem, parts of which have been previously proved in [11] and [12] (as indicated in the proof).

Theorem 4.1. *The following results hold for the surface integration problem with $F = F_{l,r,C_1}$ and $G = G_{d,l,s,C_2,c_2}$.*

1. *Let $l = 1$, so that $d = 1$ necessarily. Then*

$$e(n; \text{SURF-INT}) \asymp n^{-r}$$

and

$$\text{comp}(\varepsilon; \text{SURF-INT}) \asymp \boldsymbol{c}\ \varepsilon^{-1/r}.$$

2. *Let $l \geq 2$ and $d < l$. If $s = 1$, then there exists $\varepsilon_0 > 0$ such that*

$$e(n; \text{SURF-INT}) \geq \varepsilon_0 \qquad \forall\, n \geq 0,$$

and so

$$\text{comp}(\varepsilon; \text{SURF-INT}) = \infty \qquad \forall\, \varepsilon < \varepsilon_0.$$

However, if $s \geq 2$, then

$$e(n; \text{SURF-INT}) \asymp n^{-\min\{r,s\}/d},$$

and

$$\text{comp}(\varepsilon; \text{SURF-INT}) \asymp \boldsymbol{c}\ \varepsilon^{-d/\min\{r,s\}}.$$

3. *Suppose that $d = l \geq 2$. If $s \geq 2$, then*

$$e(n; \text{SURF-INT}) \asymp n^{-\min\{r/d, s/(d-1)\}},$$

and

$$\text{comp}(\varepsilon; \text{SURF-INT}) \asymp \boldsymbol{c}\ \varepsilon^{-\max\{d/r, (d-1)/s\}}.$$

First part of proof. The results for cases 1 and 2 were established in [11] and [12]. For case 3, the bounds

$$n^{-r/d} \preccurlyeq e(n; \text{SURF-INT}) \preccurlyeq n^{-\min\{r,s\}/d},$$

and

$$\mathbf{c}\ \varepsilon^{-d/r} \preccurlyeq \text{comp}(\varepsilon; \text{SURF-INT}) \preccurlyeq \mathbf{c}\ \varepsilon^{-d/\min\{r,s\}}$$

were proved in [11], whereas the bounds

$$n^{-s/(d-1)} \preccurlyeq e(n; \text{SURF-INT})$$

and

$$\mathbf{c}\ \varepsilon^{-(d-1)/s} \preccurlyeq \operatorname{comp}(\varepsilon; \text{SURF-INT})$$

were proved in [12].

Suppose first that $r \le s$ in case 3. Then

$$\frac{r}{d} \le \frac{s}{d} < \frac{s}{d-1},$$

and so

$$\frac{\min\{r, s\}}{d} = \frac{r}{d} = \min\left\{\frac{r}{d}, \frac{s}{d-1}\right\}.$$

Hence we find that

$$n^{-\min\{r/d, s/(d-1)\}} \preccurlyeq e(n; \text{SURF-INT}) \preccurlyeq n^{-\min\{r,s\}/d} \asymp n^{-\min\{r/d, s/(d-1)\}}, \qquad (10)$$

and so the lower and upper bounds found in Theorem 4.1 match when $r \le s$. Hence, it remains to consider case 3 for $r > s \ge 2$. $\Diamond$

We will complete the proof of Theorem 4.1 for $d = l \ge 2$ and $r > s \ge 2$ by exhibiting an algorithm whose cost is proportional to $\mathbf{c}\ n$ and whose error is proportional to $n^{-\min\{r/d, s/(d-1)\}}$. Before doing this, we reduce our problem of computing

$$\int_{g(I^d)} f\, d\sigma = \int_{I^d} f\big(g(x)\big)|(\det \nabla g)(x)|\, dx$$

to that of computing

$$S(f, g) = \int_{I^d} f\big(g(x)\big)(\det \nabla g)(x)\, dx. \qquad (11)$$

We do this by noting that for any $g \in G$, we have either $(\det \nabla g)(x) > 0$ for all $x \in I^d$ or $(\det \nabla g)(x) < 0$ for all $x \in I^d$. Hence it follows that for any $[f, g] \in F \times G$, we have

$$\int_{g(I^d)} f\, d\sigma = \pm S(f, g).$$

It only remains to determine the sign. If calculating derivatives of g were a permissible information operation, then we would calculate $(\det \nabla g)(x^*)$ at some $x^* \in I^d$, from which we would know which sign to use. However, we are not allowing derivative information. There are two approaches we can use:

1. If $s \geq 2$, then we can approximate $(\det \nabla g)(x^*)$ using difference quotients. This can be done with cost independent of n.

2. If $s = 1$, we can approximate $S(1, g)$ using the techniques in [11]. We then use the sign of our approximation as our multiplier. This can be done with cost proportional to that of the algorithm that we will present in the sequel.

Hence minimal error and optimal complexity for surface integration and for the "signed" surface integration problem of (11) are essentially the same.

Recall that one of the main tools used to prove upper bounds in [12] was [12, Lemma 4.1], which shows how we can express the volume of $g(I^d)$ as a sum of $(d-1)$-dimensional integrals. This Lemma was based on [4, Chap. 4, Theorem 3.2], which shows that a Jacobian determinant can be expressed in divergence form. We now extend [12, Lemma 4.1] to cover surface integrals.

Lemma 4.1. *Let $f \in C(I^d)$ and $g \in C^1(I^d; I^d)$. Then*

$$S(f, g) = \sum_{j=1}^{d} (-1)^{j+1} \int_{I^{d-1}} \left[(If)\big(g(x)\big) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x) \right]_{x_j=0}^{x_j=1} d\hat{x}_j,$$

where S is given by (11)*,*

$$(If)(t) = \int_0^{t_1} f(\tau, t_2, \ldots, t_d)\, d\tau, \tag{12}$$

the Jacobian determinant of the mapping

$$(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \mapsto \big(g_2(x), \ldots, g_d(x)\big)$$

is denoted by

$$\frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j},$$

and

$$d\hat{x}_j = dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_d.$$

Proof. Let

$$\mathrm{RHS}(f, g) = \sum_{j=1}^{d} (-1)^{j+1} \int_{I^{d-1}} \left[(If)\big(g(x)\big) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x) \right]_{x_j=0}^{x_j=1} d\hat{x}_j.$$

Note that $S, \mathrm{RHS}\colon C(I^d) \times C^1(I^d; I^d) \to \mathbb{R}$ are continuous nonlinear functionals, the space of polynomials is dense in $C(I^d)$, and $C^2(I^d; I^d)$ is dense in $C^1(I^d; I^d)$.

Hence, it suffices to show that $S(f, g) = \text{RHS}(f, g)$ for any polynomial f and any $g \in C^2(I^d; I^d)$. Moreover, since $S(\cdot, g)$ and $\text{RHS}(\cdot, g)$ are linear functionals for any $g \in C^2(I^d; I^d)$, it suffices to show that

$$S(\mathrm{id}^\alpha, g) = \text{RHS}(\mathrm{id}^\alpha, g), \tag{13}$$

where, for any multi-index α, we write

$$\mathrm{id}^\alpha(x) = x^\alpha \qquad \forall\, x \in I^d.$$

So, choose a multi-index α. Note that by [4, Chap. 4, Theorem 3.2], we have

$$(\det \nabla g)(x) = \sum_{j=1}^{d} (-1)^{j+1} \frac{\partial}{\partial x_j} \left[g_1(x) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x) \right]. \tag{14}$$

Hence

$$S(\mathrm{id}^\alpha, g) = A - B, \tag{15}$$

where

$$A = \sum_{j=1}^{d} (-1)^{j+1} \int_{I^d} \frac{\partial}{\partial x_j} \left[g(x)^\alpha g_1(x) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x) \right] dx$$

and

$$B = \sum_{j=1}^{d} (-1)^{j+1} \int_{I^d} \frac{\partial}{\partial x_j} \left[g(x)^\alpha \right] g_1(x) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x)\, dx.$$

From the fundamental theorem of calculus, we have

$$A = \sum_{j=1}^{d} (-1)^{j+1} \int_{I^{d-1}} \left[g_1(x)^{\alpha_1+1} g_2(x)^{\alpha_2} \ldots g_d(x)^{\alpha_d} \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x) \right]_{x_j=0}^{x_j=1} d\hat{x}_j.$$

Moreover,

$$\begin{aligned} B &= \sum_{j=1}^{d} (-1)^{j+1} \int_{I^d} \left(\sum_{i=1}^{d} \alpha_i \frac{g(x)^\alpha}{g_i(x)} \frac{\partial g_i}{\partial x_j}(x) \right) g_1(x) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x)\, dx \\ &= \int_{I^d} \sum_{i=1}^{d} \alpha_i \theta_i(x) \frac{g(x)^\alpha}{g_i(x)} g_1(x)\, dx, \end{aligned}$$

where

$$\theta_i = \sum_{j=1}^{d} (-1)^{j+1} \frac{\partial g_i}{\partial x_j} \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j} = \begin{vmatrix} \dfrac{\partial g_i}{\partial x_1} & \dfrac{\partial g_2}{\partial x_1} & \cdots & \dfrac{\partial g_d}{\partial x_1} \\[2ex] \dfrac{\partial g_i}{\partial x_2} & \dfrac{\partial g_2}{\partial x_2} & \cdots & \dfrac{\partial g_d}{\partial x_2} \\[2ex] \vdots & \vdots & \ddots & \vdots \\[2ex] \dfrac{\partial g_i}{\partial x_d} & \dfrac{\partial g_2}{\partial x_d} & \cdots & \dfrac{\partial g_d}{\partial x_d} \end{vmatrix}$$

$$= \delta_{i,1} \det \nabla g.$$

Hence

$$B = \int_{I^d} \sum_{i=1}^{d} \alpha_i \delta_{i,1} \frac{g(x)^\alpha}{g_i(x)} g_1(x) (\det \nabla g)(x)\, dx = \alpha_1 \int_{I^d} g(x)^\alpha (\det \nabla g)(x)\, dx$$

$$= \alpha_1 S(\mathrm{id}^\alpha, g).$$

Substituting this result into (15), we have

$$S(\mathrm{id}^\alpha, g) = A - \alpha_1 S(\mathrm{id}^\alpha, g).$$

Solving for $S(\mathrm{id}^\alpha, g)$, we find that

$$S(\mathrm{id}^\alpha, g) = \frac{A}{\alpha_1 + 1}$$

$$= \sum_{j=1}^{d} (-1)^{j+1} \int_{I^{d-1}} \left[\frac{1}{\alpha_1 + 1} g_1(x)^{\alpha_1 + 1} g_2(x)^{\alpha_2} \ldots g_d(x)^{\alpha_d} \right.$$

$$\left. \times \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x) \right]_{x_j=0}^{x_j=1} d\hat{x}_j.$$

Now

$$\frac{1}{\alpha_1 + 1} t_1^{\alpha_1+1} t_2^{\alpha_2} \ldots t_d^{\alpha_d} = \int_0^{t_1} \tau^{\alpha_1} t_2^{\alpha_2} \ldots t_d^{\alpha_d}\, d\tau = (I\, \mathrm{id}^\alpha)(t),$$

and so

$$\frac{1}{\alpha_1 + 1} g_1(x)^{\alpha_1+1} g_2(x)^{\alpha_2} \ldots g_d(x)^{\alpha_d} = (I\, \mathrm{id}^\alpha)\big(g(x)\big).$$

Hence

$$\frac{A}{\alpha_1 + 1} = \mathrm{RHS}(\mathrm{id}^\alpha, g),$$

which establishes (13) and, hence, the lemma. □

Next, we construct a multivariate spline space on I^d, following the approach in [11]. For a given positive integer n, let $\mathscr{Q}_n$ be a uniform grid on I^d with mesh-size $\Theta(n^{-1/d})$. Recalling that $r \geq 3$ in this part of the proof, we let $\mathscr{S}_n$ be a globally C^{r-2} tensor spline space of degree $r-1$ over $\mathscr{Q}_n$. The quasi-interpolation operator Q_n for $\mathscr{S}_n$ has the form

$$Q_n f = \sum_{j=1}^{m} \lambda_j(f) B_j \qquad \forall f \in C(I^d),$$

where $m = \dim \mathscr{S}_n \asymp n$, the functions $B_1, \ldots, B_m$ are d-fold tensor products of univariate normalized B-splines, and $\lambda_1, \ldots, \lambda_m \in [C(I^d)]^*$ satisfy

$$\lambda_j(B_i) = \delta_{i,j} \qquad (1 \leq i, j \leq m).$$

We can calculate $\{\lambda_1(f), \ldots, \lambda_m(f)\}$ for $f \in C(I^d)$ using $\Theta(n)$ evaluations of f at points in I^d. Moreover, following the same method of proof as in [7, Theorem 5.8], one can show that

$$\|f - Q_n f\|_{W^{k,\infty}(I^d)} \preccurlyeq n^{-(r-k)/d} \|f\|_{W^{r,\infty}(I^d)} \tag{16}$$

for $f \in F$ and $k \in \{0, 1, 2\}$. Recalling the definition (11) of $S(f, g)$ and using (16), along with the fact that $g(I^d) \subseteq I^d$, we find that

$$\begin{aligned} |S(f, g) - S(Q_n f, g)| &\leq \int_{g(I^d)} |f(t) - (Q_n f)(t)| \, dt \\ &= \|f - Q_n f\|_{L_1(I^d)} \preccurlyeq n^{-r/d} \|f\|_{W^{r,1}(I^d)} \preccurlyeq n^{-r/d}. \end{aligned} \tag{17}$$

It therefore remains to approximate $S(Q_n f, g)$.

Let $j \in \{1, \ldots, d\}$ and $a \in \{0, 1\}$. We define

$$S_{j,a}(v, g) = \int_{I_{j,a}^{d-1}} v(x) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x) \, d\hat{x}_j \qquad \forall\, v \in C(I_{j,a}^{d-1}), g \in G, \tag{18}$$

where

$$I_{j,a}^{d-1} = \{\, x \in I^d : x_j = a \,\}.$$

By Lemma 4.1, we have

$$S(Q_n f, g) = \sum_{j=1}^{d} (-1)^{j+1} [S_{j,1}(I Q_n f \circ, g) - S_{j,0}(I Q_n f \circ, g)],$$

where I is given by (12).

We need to approximate $S_{j,a}$ for $j \in \{1, \ldots, d\}$ and $a \in \{0, 1\}$. Similarly to what we have done above, let $\mathscr{Q}_{n,j,a}$ be a uniform grid on $I^{d-1}_{j,a}$ with mesh-size $\Theta(n^{-1/(d-1)})$. Let $\mathscr{S}_{n,j,a}$ be a globally $C^{\max\{s-2,1\}}$ tensor spline space of degree $\max\{s-1, 2\}$ over $\mathscr{Q}_{n,j,a}$. The quasi-interpolation operator $Q_{n,j,a}$ for $\mathscr{S}_{n,j,a}$ has the form

$$Q_{n,j,a}w = \sum_{j=1}^{m} \lambda_j(w) B_j \qquad \forall w \in C(I^{d-1}_{j,a}),$$

where $m = \dim \mathscr{S}_{n,j,a} \asymp n$, the functions $B_1, \ldots, B_m$ are $(d-1)$-fold tensor products of univariate normalized B-splines, and $\lambda_1, \ldots, \lambda_m \in [C(I^{d-1}_{j,a})]^*$ satisfy

$$\lambda_j(B_i) = \delta_{i,j} \qquad (1 \le i, j \le m).$$

We can calculate $\{\lambda_1(w), \ldots, \lambda_m(w)\}$ for $w \in C(I^{d-1}_{j,a})$ using $\Theta(n)$ evaluations of w at points in $I^{d-1}_{j,a}$. Moreover, we have

$$\|w - Q_{n,j,a}w\|_{W^{k,\infty}(I^{d-1}_{j,a})} \preccurlyeq n^{-(s-k)/(d-1)} \|w\|_{W^{s,\infty}(I^{d-1}_{j,a})}$$

for $w \in W^{s,\infty}(I^{d-1}_{j,a})$ and $k \in \{0, 1, 2\}$. Defining $Q_{n,j,a}$ for $\mathbb{R}^l$-valued functions componentwise, i.e.,

$$Q_{n,j,a}g = (Q_{n,j,a}g_1, \ldots, Q_{n,j,a}g_d),$$

we see that

$$\|g - Q_n g\|_{W^{k,\infty}(I^{d-1}_{j,a}; I^d)} \preccurlyeq n^{-(s-k)/(d-1)} \|g\|_{W^{s,\infty}(I^{d-1}_{j,a}; I^d)} \tag{19}$$

for $g \in W^{s,\infty}(I^{d-1}_{j,a}; I^d)$ and $k \in \{0, 1, 2\}$. Now let

$$U_{n,j,a}(v, g) = \int_{I^{d-1}_{j,a}} (Q_{n,j,a}v)(x) \frac{\partial(Q_{n,j,a}g_2, \ldots, Q_{n,j,a}g_d)}{\partial \hat{x}_j}(x)\, d\hat{x}_j \qquad \forall\, v \in C(I^{d-1}_{j,a}),\ g \in G.$$

Lemma 4.2. *Recall that $s \ge 2$. For each $j \in \{1, \ldots, d\}$ and $a \in \{0, 1\}$, we have*

$$|S_{j,a}(IQ_n f \circ g, g) - U_{n,j,a}(IQ_n f \circ g, g)| \preccurlyeq n^{-s/(d-1)} \qquad \forall\,[f, g] \in F \times G.$$

Proof. Let $[f, g] \in F \times G$. From [11], it immediately follows that for $v \in W^{2,\infty}(I^{d-1}_{j,a})$, we can approximate integrals

$$\int_{I^{d-1}_{j,a}} v\big(g(x)\big) \frac{\partial(g_2, \ldots, g_d)}{\partial \hat{x}_j}(x)\, d\hat{x}_j$$

by

$$\int_{I_{j,a}^{d-1}} \big(Q_{n,j,a}(v \circ g)\big)(x) \frac{\partial(Q_{n,j,a}g_2, \dots, Q_{n,j,a}g_d)}{\partial \hat{x}_j}(x)\, d\hat{x}_j,$$

with an error of at most $\kappa_{v,g} n^{-\min\{r,s\}/(d-1)} = \kappa_{v,g} n^{-s/(d-1)}$. Here, $\kappa_{v,g}$ is uniformly bounded in terms of $\|g\|_{W^{2,\infty}(I_{j,a}^{d-1})}$, $\|Q_{n,j,a}g\|_{W^{2,\infty}(I_{j,a}^{d-1})}$, and $\|v\|_{W^{2,\infty}(I_{j,a}^{d-1})}$. It is easy to see that the proof of this result also establishes that

$$|S_{j,a}(v, g) - U_{n,j,a}(v, g)| \le \kappa_{v,g} n^{-s/(d-1)}, \tag{20}$$

where $\kappa_{v,g}$ is once again uniformly bounded in terms of the same norms of g, $Q_n g$, and v as before.

Now Q_n and I are uniformly bounded operators on $W^{2,\infty}(I_{j,a}^{d-1})$. Since $f \in C^r(I^d)$ and $g \in C^s(I^d; I^d)$, and since (16) holds for $k \in \{0, 1, 2\}$, we have $I Q_n f \circ g \in W^{2,\infty}(I_{j,a}^{d-1})$ with uniformly bounded norm. Letting $v = I Q_n f \circ g$ in (20), we get the desired result. □

We finally define the algorithm U_n approximating S as

$$U_n(f, g) = \sum_{j=1}^{d} (-1)^{j+1} [U_{n,j,1}(I Q_n f \circ Q_{n,j,1} g, g) - U_{n,j,0}(I Q_n f \circ Q_{n,j,0} g, g)].$$

Note that for any $j \in \{1, \dots, d\}$ and $a \in \{0, 1\}$, the functions $Q_n f$ and $I Q_n f$ are polynomial of fixed degree (depending only on d, r and s) on each $K \in \mathscr{Q}_{n,j,a}$. Hence, the same is true for the function $I Q_n f \circ Q_{n,j,a} g$. Thus for any $x \in I_{j,a}^{d-1}$, we can calculate $(I Q_n f \circ Q_{n,j,a} g)(x)$ with cost $\Theta(\mathbf{c})$. Now we can express $U_n(f, g)$ as the sum of integrals over each $K \in \mathscr{Q}_{n,ja}$. Each such integral can be computed exactly with cost $\Theta(c)$, since its integrand is a polynomial of fixed degree. Since we have $\Theta(n)$ such integrals, the total cost of computing $U_n(f, g)$ satisfies

$$\operatorname{cost} U_n \asymp \mathbf{c}\, n.$$

We are now ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1 *(conclusion).* Recall that we have $d = l \ge 2$ and $r > s \ge 2$. It suffices to prove that

$$|S(f, g) - U_n(f, g)| \preccurlyeq n^{-\min\{r/d, s/(d-1)\}}.$$

Let

$$\tilde{U}_n(f, g) = \sum_{j=1}^{d} (-1)^{j+1} [U_{n,j,1}(I Q_n f \circ g, g) - U_{n,j,0}(I Q_n f \circ g, g)].$$

Using (17) and Lemma 4.2, we find that

$$\begin{aligned} |S(f, g) - \tilde{U}_n(f, g)| &\le |S(f, g) - S(Q_n f, g)| + |S(Q_n f, g) - \tilde{U}_n(f, g)| \\ &\preccurlyeq n^{-r/d} + n^{-s/(d-1)} \preccurlyeq n^{-\min\{r/d, s/(d-1)\}}. \end{aligned} \tag{21}$$

Suppose we can show that

$$|U_{n,j,a}(I Q_n f \circ g, g) - U_{n,j,a}(I Q_n f \circ Q_{n,j,a} g, g)| \preccurlyeq n^{-s/(d-1)} \tag{22}$$

for $j \in \{1, \dots, d\}$ and $a \in \{0, 1\}$. From the definitions of U_n and $\tilde{U}_n$, it will then follow that

$$|U_n(f, g) - U_n(f, g)| \preccurlyeq n^{-s/(d-1)},$$

which, together with (21), yields the desired result.

So, for $j \in \{1, \dots, d\}$ and $a \in \{0, 1\}$, we wish to prove (22). It will simplify the notation in what follows if we write $\bar{f} = Q_n f$ and $\bar{g} = Q_{n,j,a} g$.

Let $x \in I_{j,a}^{d-1}$. We have

$$(I \bar{f} \circ g)(x) - I(\bar{f} \circ \bar{g})(x) = A_1(x) - A_2(x),$$

where

$$A_1(x) = \int_0^{g_1(x)} \left[\bar{f}\big(\tau, g_2(x), \dots, g_d(x)\big) - \bar{f}\big(\tau, \bar{g}_2(x), \dots, \bar{g}_d(x)\big) \right] d\tau$$

and

$$A_2(x) = \int_{g_1(x)}^{\bar{g}_1(x)} \bar{f}\big(\tau, \bar{g}_2(x), \dots, \bar{g}_d(x)\big)\, d\tau.$$

Hence

$$|U_{n,j,a}(I \bar{f} \circ g, g) - U_{n,j,a}(I \bar{f} \circ \bar{g}, g)| \le |B_1| + |B_2|, \tag{23}$$

where

$$B_i = \int_{I_{j,a}^{d-1}} (Q_{n,j,a} A_i)(x) \frac{\partial(\bar{g}_2, \dots, \bar{g}_d)}{\partial \hat{x}_j}(x)\, d\hat{x}_j \qquad \text{for } i \in \{1, 2\}.$$

We first estimate $|B_1|$, finding

$$|B_1| \le \|Q_{n,j,a}A_1\|_{L_1(I_{j,a}^{d-1})} \int_{I_{j,a}^{d-1}} \left| \frac{\partial(\bar g_2,\dots,\bar g_d)}{\partial \hat x_j}(x) \right| d\hat x_j \preccurlyeq \|A_1\|_{L_1(I_{j,a}^{d-1})}$$
$$\le \int_{I_{j,a}^{d-1}} \int_0^1 \left| \bar f\bigl(\tau, g_2(x),\dots,g_d(x)\bigr) - \bar f\bigl(\tau, \bar g_2(x),\dots,\bar g_d(x)\bigr)\right| d\tau\, d\hat x_j.$$

Since $r \ge 2$, the bound (16) yields that $\bar f$ is a Lipschitz function whose Lipschitz constant is uniformly bounded. Using (19), we thus have

$$|B_1| \preccurlyeq \|g-\bar g\|_{L_\infty(I_{j,a}^{d-1};I^d)} \preccurlyeq n^{-s/(d-1)}. \tag{24}$$

We next estimate $|B_2|$. We have

$$|B_2| \preccurlyeq \|Q_{n,j,a}A_2\|_{L_\infty(I_{j,a}^{d-1})} \preccurlyeq \left\| \int_{g_1(\cdot)}^{\bar g_1(\cdot)} \bar f\bigl(\tau, \bar g_2(\cdot),\dots,\bar g_d(\cdot)\bigr)\, d\tau \right\|_{L_\infty(I_{j,a}^{d-1})}.$$

Now for $x \in I_{j,a}^{d-1}$, we have $f\bigl(\cdot, \bar g_2(\cdot),\dots,\bar g_d(\cdot)\bigr) \in C(I)$ with uniformly bounded $C(I)$-norm, and so

$$\left| \int_{g_1(x)}^{\bar g_1(x)} \bar f\bigl(\tau, \bar g_2(x),\dots,\bar g_d(x)\bigr)\, d\tau \right| \preccurlyeq |g_1(x)-\bar g_1(x)|.$$

Hence

$$\left\| \int_{g_1(\cdot)}^{\bar g_1(\cdot)} \bar f\bigl(\tau, \bar g_2(\cdot),\dots,\bar g_d(\cdot)\bigr)\, d\tau \right\|_{L_\infty(I_{j,a}^{d-1})} \preccurlyeq \|g_1-\bar g_1\|_{L_\infty(I_{j,a}^{d-1})},$$

and so

$$|B_2| \preccurlyeq n^{-s/(d-1)}. \tag{25}$$

Using the bounds (24) and (25) in (23), we find that (22) holds, which establishes the desired bound, and completes the proof. □

Hence we have found the nth minimal error and the ε-complexity for all instances of the surface integration problem, with the exception of the case $d = l \ge 2$ and $s = 1$. What happens in this "minimal smoothness" case?

We give a partial result, which shows that the results in case 3 of Theorem 4.1 hold when $d = 2$:

Theorem 4.2. *Let $d = l = 2$, $r \ge 1$, and $s = 1$. Then*

$$e(n; \text{SURF-INT}) \asymp n^{-\min\{r/d,s/(d-1)\}} \asymp n^{-\min\{r/2,1\}},$$

and

$$\mathrm{comp}(\varepsilon; \text{SURF-INT}) \asymp \boldsymbol{c}\ \varepsilon^{-\max\{d/r,(d-1)/s\}} \asymp \boldsymbol{c}\ \varepsilon^{-\max\{2/r,1\}}.$$

Proof. Since the desired lower bounds are found in [12], we need only prove the upper bounds. From [11], we also know that

$$e(n;\ \text{SURF-INT}) \preccurlyeq n^{-\min\{r,1\}/2}.$$

Hence, we need only consider the case $r \geq 2$. Clearly $\min\{\frac{1}{2}r, 1\} = 1$ in this case. Hence, it suffices to exhibit an algorithm U_n with

$$\operatorname{cost} U_n \asymp \mathbf{c}\ n \qquad \text{and} \qquad e(U_n) \leq n^{-1}. \tag{26}$$

By Lemma 4.1, we have

$$S(f, g) = \sum_{j=1}^{2} \sum_{a=0}^{1} (-1)^{j+1-a} S_{j,a}(If \circ g, g)$$

where $S_{j,a}$ is defined by (18), so that

$$S_{j,a}(If \circ g, g) = \int_0^1 \left[(If)\big(g(x)\big) \frac{\partial g_2}{\partial x_{3-j}}(x) \right]_{x_j = a} dx_{3-j}.$$

Each $S_{j,a}(If \circ g, g)$ is a Stieltjes integral of the form $\int_0^1 w(\xi)\, du(\xi)$, where $w \in W^{1,\infty}(I)$ and $u \in C^1(I)$. By a straightforward modification of the techniques in [10], there exists an approximation $S_{n,j,a}(If \circ, g)$ of $S_{j,a}(If \circ g, g)$ such that

$$\operatorname{cost} S_{n,j,a} \asymp \mathbf{c}\ n$$

and

$$|S_{j,a}(If \circ g, g) - S_{n,j,a}(If \circ g, g)| \preccurlyeq n^{-1}.$$

Letting

$$U_n(f, g) = \sum_{j=1}^{2} \sum_{a=0}^{1} (-1)^{j+1-a} S_{n,j,a}(If \circ g, g),$$

we see that (26) holds, as required. □

Remark. So far, we have found tight bounds on the minimal error and complexity for all cases, except for the case $d = l \geq 3$ with $r \geq s = 1$. Note that we may use the techniques of [12] to see that if $d = l$ and $r \geq s = 1$, then

$$n^{-\min\{r/d, 1/(d-1)\}} \preccurlyeq e(n;\ \text{SURF-INT}) \preccurlyeq n^{-\min\{r/d, 2/(d(d-1))\}}$$

and

$$\mathbf{c}\ \varepsilon^{-\max\{d/r, d-1\}} \leq \operatorname{comp}(\varepsilon;\ \text{SURF-INT}) \preccurlyeq \mathbf{c}\ \varepsilon^{-\max\{d/r, d(d-1)/2\}}.$$

Of course, these bounds tell us that our problem certainly has finite complexity. If $r = s = 1$ and $d = 3$, these bounds become tight and we have

$$e(n;\ \text{SURF-INT}) \asymp n^{-1/3}$$

and

$$\text{comp}(\varepsilon;\ \text{SURF-INT}) \asymp \mathbf{c}\ \varepsilon^{-3}. \quad \square$$

We may summarize the results of this section as

Theorem 4.3. *Consider the surface integration problem with $F = F_{l,r,C_1}$ and $G = G_{d,l,s,C_2,c_2}$. Let*

$$\nu = \min\left\{\frac{r}{d}, \frac{s - \delta_{s,1}(1 - \delta_{d,l})}{\min\{d, l-1\}}\right\}.$$

Suppose that any of the following hold:

- $r = 0$*,*
- $d < l$*,*
- $d = l = 1$*,*
- $d = l \geq 2$ *and* $s \geq 2$*,*
- $d = l = 2$ *and* $s = 1$*, or*
- $d = l = 3$ *and* $r = s = 1$.

Then

$$e(n;\ \text{SURF-INT}) \asymp n^{-\nu}$$

and

$$\text{comp}(\varepsilon;\ \text{SURF-INT}) \asymp \mathbf{c}\ \varepsilon^{-1/\nu}. \quad \square$$

We conjecture that the conclusion of Theorem 4.1 holds for all values of d, l, r, and s.

5 Approximation and integration over smooth surfaces

In the previous sections, we have determined the complexity of approximation and integration over images of the unit cube. Such images necessarily have corners. What can we say about approximation and integration over smooth surfaces?

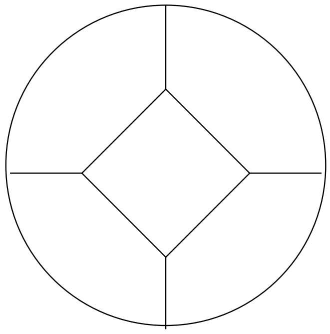

Figure 1: Cellulation of a smooth region

We first note that our results concerning surface approximation and integration may easily be extended to *oriented cellulated regions*. Roughly speaking, these regions are unions

$$\Sigma_g = \bigcup_{i=1}^{k} g^{[i]}(I^d), \tag{27}$$

with $g^{[1]}(I^d), \ldots, g^{[k]}(I^d)$ having disjoint interiors; see [5, pp. 369–370] for a precise definition. Examples of oriented cellulated regions include d-dimensional balls (see Figure 1 for $d = 2$) and spheres, as well as more general smooth regions.

Let us say that an oriented cellulated region is an *oriented k-cellulated region* if it is a cellulated region of the form (27). Suppose now that for a given $\bar{k} \geq 1$, our class of surfaces consists of oriented k-cellulated regions using maps $g^{[1]}, \ldots, g^{[k]}$ from $G = G_{d,l,s,C_2,c_2}$, where $k \in \{1, \ldots, \bar{k}\}$. Let our class of functions to be approximated or integrated once again be given by $F = F_{l,r,C_1}$. Our surface approximation problem now consists of finding, for $k \in \{1, \ldots, \bar{k}\}$ and $[f, g^{[1]}, \ldots, g^{[k]}] \in F \times G^k$, a k-tuple $U = (U_1, \ldots, U_k)$ of functions $U_1(f, g^{[1]}), \ldots, U_k(f, g^{[k]}) \in L_q(I^d)$ such that

$$e(U; \text{SURF-APP}) :=$$

$$\max_{1 \leq k \leq \bar{k}} \sup_{[f, g^{[1]}, \ldots, g^{[k]}] \in F \times G^k} \left(\sum_{i=1}^{k} \|\sigma_{g^{[i]}}^{1/q} [f \circ g^{[i]} - U(f, g^{[i]})]\|_{L_q(I^d)}^q \right)^{1/q}$$

(with the usual modification when $q = \infty$) is small.

Our surface integration problem now consists of finding, for $k \in \{1, \ldots, \bar{k}\}$

and $[f, g^{[1]}, \dots, g^{k]}] \in F \times G^k$, a number $U(f, g^{[1]}, \dots, g^{[k]}) \in \mathbb{R}$ such that

$$e(U; \text{SURF-INT}) :=$$

$$\max_{1 \le i \le k} \sup_{[f, g_1, \dots, g_k] \in F \times G^k} \left| \sum_{i=1}^{k} \int_{g^{[i]}(I^d)} f \, d\sigma - U(f, g^{[1]}, \dots, g^{k]}) \right|$$

is small.

Theorems 3.1, 4.1, and 4.2 hold for this variant of the surface approximation and surface integration problems. Indeed, for upper bounds, it suffices to approximate f or integrate f over each $g^{[i]}(I^d)$. For lower bounds, it is enough to note that the nth minimal error increases with $\bar{k}$ and to use the lower bounds of Sections 3 and 4, which hold for the case $\bar{k} = 1$.

The main problem with this approach is that it requires us to specify the k maps $g^{[1]}, \dots, g^{[k]}$ that define each of our surfaces. This may not be so easy to do in practice, even for simple regions such as d-dimensional balls.

We may overcome the difficulty of defining such maps $g^{[1]}, \dots, g^{[k]}$ by adopting another approach, which we illustrate for the case of surfaces that are images of the Euclidean unit ball B_d in $\mathbb{R}^l$. Then for a function $g\colon B_d \to \mathbb{R}^l$, we can discuss approximation and integration for functions $f\colon g(B_d) \to \mathbb{R}$. We shall show that the results for the approximation and integration problems over images of balls are essentially the same as those over images of cubes.

We first note that when $d = 1$, we have $B_1 = [-1, 1]$. Hence we see that, ignoring constant factors, the minimal error and complexity for approximation and integration over images of balls are the same as for these problems over images of cubes when $d = 1$.

Thus, we restrict our attention to the case $d \ge 2$. The key here will be to note that B_d is the image of I^d under spherical coordinates. The spherical coordinate map T has the form

$$\begin{aligned}
[T(x)]_i &= x_1 \left(\prod_{j=2}^{i} \sin \pi x_j \right) \cos \pi x_{i+1} \qquad (1 \le i \le d-2), \\
[T(x)]_{d-1} &= x_1 \left(\prod_{j=2}^{d-1} \sin \pi x_j \right) \cos 2\pi x_d, \\
[T(x)]_d &= x_1 \left(\prod_{j=2}^{d-1} \sin \pi x_j \right) \sin 2\pi x_d.
\end{aligned}$$

Note that T maps I^d onto B_d, but the mapping is *not* an injection. It is easy to see that

$$\sigma_T(x) = |(\det \nabla T)(x)| = 2(\pi x_1)^{d-1} \prod_{j=2}^{d-1} |\sin \pi x_j|^{d-j}. \tag{28}$$

See, e.g., [5, p. 268] for further details on spherical coordinates.

We define surface integrals over injective images of B_d in the obvious way as

$$\int_{g(B_d)} f\, d\sigma = \int_{B_d} (f \circ g)\, \sigma_g \equiv \int_{B_d} f\bigl(g(y)\bigr)\, \sigma_g(y)\, dy,$$

where σ_g is defined as in Section 2. By [5, pp. 255-257], we can use the change of variables formula

$$\int_{B_d} h(y)\, dy = \int_{I^d} h\bigl(T(x)\bigr)\, \sigma_T(x)\, dx$$

to find that

$$\int_{g(B_d)} f\, d\sigma = \int_{I^d} f\bigl[g\bigl(T(x)\bigr)\bigr]\, \sigma_g\bigl(T(x)\bigr)\, \sigma_T(x)\, dx,$$

where $\sigma_T(x)$ is given by (28).

Next, we define our classes F and G of functions.

For a positive C_1, and for $r \geq 0$, we first define $F = F^{\circ}_{l,r,C_1}$ as the class of r times continuously differentiable functions $f \in C^r(B_l)$ that satisfy

$$\|f\|_{C^r(B_l)} \leq C_1 \qquad \forall f \in F.$$

Here, B_l is the Euclidean unit ball in $\mathbb{R}^d$.

For positive C_2 and c_2 and for $s \geq 1$, we define $G = G^{\circ}_{d,l,s,C_2,c_2}$ as the class of s times continuously differentiable functions $g \in C^s(B_d; B_l)$ that satisfy

$$\|g\|_{C^s(B_d;B_l)} \leq C_2 \qquad \text{and} \qquad \min_{x \in B_d} \sigma_g(x) \geq c_2.$$

where σ_g is as defined in Section 2.

We can now define our two problems as in Section 2, but with the obvious changes. The surface approximation problem SURF-APP over B_d consists of finding, for $[f, g] \in F \times G$, a function $U(f, g) \in L_q(B_d)$ such that

$$e(U; \text{SURF-APP}; B_d) := \sup_{[f,g] \in F \times G} \|\sigma_g^{1/q}[f \circ g - U(f, g)]\|_{L_q(B_d)}$$

is small. The surface integration problem SURF-INT over B_d consists of finding, for $[f, g] \in F \times G$, an approximation $U(f, g) \in \mathbb{R}$ such that

$$e(U; \text{SURF-INT}; B_d) := \sup_{[f,g] \in F \times G} \left| \int_{g(B_d)} f\, d\sigma - U(f, g) \right|$$

is small.

With these definitions in hand, we now specify concepts such as information, algorithm, minimal error, cost, complexity, and optimality for SURF-APP and SURF-INT over B_d as was done in Section 2 for SURF-APP and SURF-INT over I^d. Note that since $r \geq 0$ and $s \geq 1$, the surface integration and approximation problems, as well as standard information, are well-defined.

Now that we have defined SURF-APP and SURF-INT over balls, we can show that the results for these problems over balls are essentially the same as for cubes.

We first treat the surface approximation problem.

Theorem 5.1. *Consider the surface approximation problem with $F = F^{\circ}_{l,r,C_1}$ and $G = G^{\circ}_{d,l,s,C_2,c_2}$. Let*

$$\mu = \frac{\min\{r, s\}}{d}.$$

Then

$$e(n; \text{SURF-APP}; B_d) \asymp e(n; \text{SURF-APP}; I^d) \asymp n^{-\mu}$$

and

$$\text{comp}(\varepsilon; \text{SURF-APP}; B_d) \asymp \text{comp}(\varepsilon; \text{SURF-APP}; I^d) \asymp \boldsymbol{c}\ \varepsilon^{-1/\mu}.$$

Proof. We first prove the lower bounds

$$e(n; \text{SURF-APP}; B_d) \succcurlyeq e(n; \text{SURF-APP}; I^d) \tag{29}$$

and

$$\text{comp}(\varepsilon; \text{SURF-APP}; B_d) \succcurlyeq \text{comp}(\varepsilon; \text{SURF-APP}; I^d).$$

Clearly, it suffices to prove (29). Let U be an algorithm for SURF-APP over B_d using information of cardinality at most n. The inclusion $d^{-1/2}I_d \subseteq B_d$ tells us that

$$\begin{aligned} e(U; \text{SURF-APP}; B_d) &= \sup_{[f,g]\in F\times G} \|\sigma_g^{1/q}[f \circ g - U(f,g)]\|_{L_q(B_d)} \\ &\geq \sup_{[f,g]\in F\times G} \|\sigma_g^{1/q}[f \circ g - U(f,g)]\|_{L_q(d^{-1/2}I_d)} \\ &= e(U; \text{SURF-APP}; d^{-1/2}I_d). \end{aligned}$$

Since U is an arbitrary algorithm using information of cardinality at most n, we see that

$$e(n; \text{SURF-APP}; B_d) \geq e(n; \text{SURF-APP}; d^{-1/2}I_d).$$

Since

$$e(n; \text{SURF-APP}; d^{-1/2}I_d) \asymp e(n; \text{SURF-APP}; I^d),$$

the desired lower bound (29) follows immediately.

We only need to prove the upper bounds

$$e(n;\,\text{SURF-APP};\,B_d) \preccurlyeq e(n;\,\text{SURF-APP};\,I^d)$$

and

$$\operatorname{comp}(\varepsilon;\,\text{SURF-APP};\,B_d) \preccurlyeq \operatorname{comp}(\varepsilon;\,\text{SURF-APP};\,I^d).$$

Let $[f,g] \in F \times G$. As in the proof of Lemma 3.1, we find that $f \circ g \circ T \in C^{\min\{r,s\}}(I^d)$ with uniformly bounded norm. As in the proof of Lemma 3.2, there exists an algorithm V_n such that

$$\|f \circ g \circ T - V_n(f \circ g \circ T)\|_{L_q(I^d)} \preccurlyeq n^{-\mu} \qquad \text{and} \qquad \operatorname{cost} V_n \asymp \mathbf{c}\, n.$$

Let

$$U_n(f,g) = V_n(f \circ g \circ T).$$

Clearly $\operatorname{cost} U_n \asymp \mathbf{c}\, n$. It only remains to show that $e(U_n) \preccurlyeq n^{-\mu}$. Suppose that $q < \infty$. Then

$$\begin{aligned}
&e(U_n,\,\text{SURF-APP};\,B_d)\\
&\quad= \sup_{[f,g]\in F\times G} \left[\int_{I^d} \sigma_g\bigl(T(x)\bigr)\,|(f\circ g\circ T)(x) - V_n(f,g)(x)|^q\,\sigma_T(x)\,dx\right]^{1/q}\\
&\quad\preccurlyeq \sup_{[f,g]\in F\times G} \|f\circ g\circ T - V_n(f,g)\|_{L_q(I^d)} \preccurlyeq n^{-\mu}.
\end{aligned}$$

The case $q = \infty$ is handled in the usual way, completing the proof. ☐

We now turn to surface integration over B_d.

Theorem 5.2. *Consider the surface integration problem with $F = F^{\circ}_{l,r,C_1}$ and $G = G^{\circ}_{d,l,s,C_2,c_2}$. Let*

$$\nu = \min\left\{\frac{r}{d},\, \frac{s - \delta_{s,1}(1-\delta_{d,l})}{\min\{d, l-1\}}\right\}.$$

Suppose that any of the following hold:

- *$r = 0$,*
- *$d = l = 1$,*
- *$d < l$,*
- *$d = l \geq 2$ and $s \geq 2$,*

- $d = l = 2$ *and* $s = 1$, *or*
- $d = l = 3$ *and* $r = s = 1$.

Then

$$e(n;\ \text{SURF-INT};\ B_d) \asymp e(n;\ \text{SURF-INT};\ I^d) \asymp n^{-\nu}$$

and

$$\mathrm{comp}(\varepsilon;\ \text{SURF-INT};\ B_d) \asymp \mathrm{comp}(\varepsilon;\ \text{SURF-INT};\ I^d) \asymp \boldsymbol{c}\ \varepsilon^{-1/\nu}.$$

Proof. We first prove the lower bounds

$$e(n;\ \text{SURF-INT};\ B_d) \succcurlyeq e(n;\ \text{SURF-INT};\ I^d) \tag{30}$$

and

$$\mathrm{comp}(\varepsilon;\ \text{SURF-INT};\ B_d) \succcurlyeq \mathrm{comp}(\varepsilon;\ \text{SURF-INT};\ I^d).$$

It suffices to prove (30). We recall that integrals of the form

$$\int_{I^d} f(x)\,dx \qquad \text{with } f \geq 0$$

or the form

$$\int_{I^d} [\sigma_{\tilde{g}}(x) - \sigma_g(x)]\,dx \qquad \text{with } \sigma_{\tilde{g}} - \sigma_g \geq 0.$$

were used in [11] and [12] to establish lower bounds on $e(n;\ \text{SURF-INT};\ I^d)$. Using the same techniques, we can establish lower bounds on $e(n;\ \text{SURF-INT};\ d^{-1/2} I^d)$ as integrals of the form

$$\int_{d^{-1/2} I^d} f(y)\,dy \qquad \text{with } f \geq 0$$

or the form

$$\int_{d^{-1/2} I^d} [\sigma_{\tilde{g}}(y) - \sigma_g(y)]\,dy \qquad \text{with } \sigma_{\tilde{g}} - \sigma_g \geq 0.$$

Since $d^{-1/2} I^d \subseteq B_d$, we have

$$\int_{B_d} f(y)\,dy \geq \int_{d^{-1/2} I_d} f(y)\,dy \qquad \text{if } f \geq 0$$

and

$$\int_{B_d} [\sigma_{\tilde{g}}(y) - \sigma_g(y)]\,dy \geq \int_{d^{-1/2} I_d} [\sigma_{\tilde{g}}(y) - \sigma_g(y)]\,dy \qquad \text{if } \sigma_{\tilde{g}} - \sigma_g \geq 0.$$

Using these last two relations, along with

$$e(n; \text{SURF-INT}; d^{-1/2} I_d) \asymp e(n; \text{SURF-INT}; I^d),$$

we get the desired result (30).

For the upper bounds, we will show that a simple modification of the known optimal algorithms for SURF-INT over I^d yields optimal algorithms for SURF-INT over B_d. The cases $d < l$ and $d = l$ need to be handled separately. For the sake of expository simplicity, we shall only give the details for the case $d < l$, the changes for the case $d = l$ being analogous.

Since the nth minimal error is infinite for $s = 1$, we need only consider the case $s \geq 2$. If $r \leq s - 1$, then we can follow the approach at the top of page 458 of [11] to see that we can compute an approximation $U_n(f, g)$ of $\int_{g(I^d)} f \, d\sigma$ for which

$$\left| \int_{g(I^d)} f \, d\sigma - U_n(f, g) \right| \preccurlyeq n^{-\min\{r,s\}/d} \qquad \text{and} \qquad \text{cost}(U_n) \asymp \mathbf{c}\, n.$$

Indeed, let $v = (f \circ g \circ T)(\sigma_g \circ T)\sigma_T$, so that

$$\int_{g(B^d)} f \, d\sigma = \int_{I_d} v.$$

Clearly, $v \in C^r(I^d)$, with uniformly bounded norm. From [6, p. 36]), we can calculate an approximation $I_n(v)$ at cost $O(\mathbf{c}\, n)$, for which

$$\left| \int_{B_d} v(x) \, dx - I_n(v) \right| \preccurlyeq \|v\|_{C^r(B_d)} \, n^{-r/d} \asymp \|v\|_{C^r(B_d)} \, n^{-\min\{r,s\}/d}.$$

Taking

$$U_n(f, g) = I_n(v),$$

we see that

$$e(U_n) \asymp n^{-\min\{r,s\}/d} \qquad \text{and} \qquad \text{cost}(U_n) \asymp \mathbf{c}\, n,$$

as required.

So it suffices to consider only the case $r \geq s$. Let $\mathcal{Q}_n$ be a uniform grid on I^d with meshsize $\Theta(n^{-1/d})$. Let $\mathcal{S}_n$ be a globally $C^{\max\{s-2,1\}}$ tensor spline space of degree $\max\{s-1, 2\}$ over $\mathcal{Q}_n$, whose ($\mathbb{R}$- or $\mathbb{R}^d$-valued, depending on context) quasi-interpolation operator is denoted by Q_n.

Following the approach in [11], for positive ξ and η, we have the expansion

$$\sqrt{\xi} = R_s(\xi, \eta) + \Theta\big((\xi - \eta)^s\big),$$

where

$$R_s(\xi, \eta) = \sqrt{\eta} + \sum_{t=1}^{s-1} \beta_t(\eta)(\xi - \eta)^t$$

with

$$\beta_t = \frac{1}{t!}\left(\frac{d}{d\xi}\right)^t \xi^{1/2}\bigg|_{\xi=\eta} = \frac{1}{\eta^{(2t-1)/2}}\binom{t-\frac{3}{2}}{t} \qquad (1 \le t \le s-1).$$

We will use this formula to approximate

$$\sigma_{Q_n g}\big(T(x)\big) = \sqrt{\det A_{Q_n g}\big(T(x)\big)}.$$

We now define our algorithm for the case $r \ge s$ as

$$U_n(f, g) = \sum_{K \in \mathscr{Q}_n} U_{n,K}(f, g),$$

where

$$U_{n,K}(f, g) = \int_K \big(Q_n(f \circ g)\big)\big(T(x)\big) \cdot R_s\Big(\det A_{Q_n g}\big(T(x)\big), \det A_{Q_n g}(y^{(K)})\Big) \cdot \sigma_T(x)\, dx$$

for each subcube $K \in \mathscr{Q}_n$. Here, $y^{(K)} = T(x^{(K)})$, where $x^{(K)}$ is any evaluation point in K; for example, it might be chosen as the center of K.

We claim that $\operatorname{cost} U_n$ is $\Theta(\mathbf{c}\, n)$. Indeed, $\det A_{Q_n g}(y)$ is a polynomial in y. Hence, $\det A_{Q_n g}\big(T(x)\big)$ is a sum of terms that are products of powers of x_1 and powers of sines and cosines of $x_2, \dots, x_d$. Moreover $Q_n(f \circ g)$ is polynomial on each $K \in \mathscr{Q}_n$, and so $Q_n(f \circ g)\big(T(x)\big)$ is a sum of terms that are products of powers of x_1 and sines and cosines of $x_2, \dots, x_d$ on each $K \in \mathscr{Q}_n$. Since $\sigma_T(x)$ is also a product of powers of x_1 and sines and cosines of $x_2, \dots, x_d$, the integrand appearing in the definition of $U_{n,K}(f, g)$ has a closed form antiderivative. Hence $U_{n,K}(f, g)$ can be computed at constant cost, once we have computed the necessary function values. Thus

$$\operatorname{cost} U_n \asymp \mathbf{c}\, n + |\mathscr{Q}_n| \cdot \max_{K \in \mathscr{Q}_n} \operatorname{cost} U_{n,K}(f, g) \asymp \mathbf{c}\, n,$$

as claimed.

Since $r \ge s$, we need only show that $e(U_n) \preccurlyeq n^{-s/d}$. Let

$$\begin{aligned} V_n(f, g) &= \int_{B_d} \big(Q_n(f \circ g)\big)(y)\, \sigma_{Q_n g}(y)\, dy \\ &= \int_{I^d} \big(Q_n(f \circ g)\big)\big(T(x)\big)\, \sigma_{Q_n g}\big(T(x)\big)\, \sigma_T(x)\, dx. \end{aligned}$$

By the triangle inequality, it suffices to show that

$$\left| \int_{g(B_d)} f\, d\sigma - V_n(f,g) \right| \preccurlyeq n^{-s/d} \tag{31}$$

and

$$|U_n(f,g) - V_n(f,g)| \preccurlyeq n^{-s/d}. \tag{32}$$

The proof of (31) is essentially the same as that of [11, Theorem 4.3]. The major distinction is that we integrate over B_d instead of over I^d. This means that the integrations by parts that led to [11, equation (19)] need to be in the form

$$\int_{B_d} \partial_i \omega = \int_{\partial B_d} \omega\, n_i$$

(where n_i is the ith component of the outward unit normal to ∂B_d) appropriate for integration over balls, rather than cubes.

It remains to prove (32). For simplicity, let $h = f \circ g$ and use overbars to denote interpolants, so that $\bar{g} = Q_n g$ and $\bar{h} = Q_n h = Q_n(f \circ g)$. Now

$$|V_n(f,g) - U_n(f,g)| \preccurlyeq \sum_{K \in \mathcal{Q}_n} \int_K \left|\bar{h}\big(T(x)\big)\right| \left|\det A_{\bar{g}}\big(T(x)\big) - \det A_{\bar{g}}\big(y^{(K)}\big)\right|^s \sigma_T(x)\, dx.$$

Since $\det A_{\bar{g}}$ has a uniformly bounded first derivative and T is Lipschitz, we have

$$\begin{aligned}\left|\det A_{\bar{g}}\big(T(x)\big) - \det A_{\bar{g}}\big(y^{(K)}\big)\right| &\preccurlyeq \|T(x) - y^{(K)}\|_{\ell_\infty(\mathbb{R}^d)} \\ &= \|T(x) - T(x^{(K)})\|_{\ell_\infty(\mathbb{R}^d)} \\ &\preccurlyeq \|x - x^{(K)}\|_{\ell_\infty(\mathbb{R}^d)} \asymp n^{-1/d}.\end{aligned}$$

From the previous two inequalities, we obtain

$$\begin{aligned}|V_n(f,g) - U_n(f,g)| &\preccurlyeq n^{-s/d} \sum_{K \in \mathcal{Q}_n} \int_K \left|\bar{h}\big(T(x)\big)\right| \sigma_T(x)\, dx \\ &= n^{-s/d} \int_{B_d} |\bar{h}(y)|\, dy \\ &\le n^{-s/d} \|\bar{h}\|_{L_1(B_d)} \preccurlyeq n^{-s/d}.\end{aligned}$$

This establishes (32), completing the proof of the theorem. □

We conjecture that the conclusion of Theorem 5.2 holds for all values of d, l, r, and s.

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