Graph Imbeddings and Overlap Matrices (Preliminary Report)

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(PRELIMINARY REPORT)

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ABSTRACT

Mohar has shown an interesting relationship between graph imbeddings and certain boolean matrices. In this paper, we show some interesting properties of this kind of matrices. Using these properties, we give the distributions of nonorietable imbeddings of several interesting infinite families of graphs, including cobblestone paths, closed-end ladders for which the distributions of orientable imbeddings are known.

1 Introduction

Gross and Furst [GF] have introduced a hierarchy of genus-respecting partitions of the set of imbeddings of a graph into a closed orientable surface. Chen and Gross [CG] have generalized the idea to include the distributions of the set of imbeddings of a graph into a closed nonorientable surface. Moreover, the distributions of orientable imbeddings of several interesting infinite families of graphs have been computed (see [FGS], [GRT] and [Mc]) by deriving and solving proper recurrence relations. However, as pointed out by Chen and Gross [CG], these techniques seem not proper for computing the distributions of nonorientable imbeddings of graphs because the interaction between orientable and nonorietable imbeddings gives us much more complicated recurrence relations which seem not easy to solve. This has forced us to look for new techniques for computing distributions of nonorientable imbeddings of graphs.

Mohar [Mo] has shown a very interesting relationship between graph imbeddings and so called overlap matrices. He showed that given a rotation system $R(G)$ of a graph G, we can construct a boolean matrix $M(R)$ such that the genus (if $R(G)$ is an orientable imbedding) or the crosscap number (if $R(G)$ is a nonorientable imbedding) of $R(G)$ is closely related to the rank of $M(R)$.

More formally, let T be a spanning tree of a graph G and let $R(G)$ be a rotation system of G such that all edges of T have type-0. Let the co-tree edges of G (w.r.t. T) be $e_1, e_2, \cdots, e_m$. We construct a $m \times m$ symmetric matrix $M(R)$ over $GF(2)$ as follows (call it *the overlap matrix corresponding to* $R(G)$): the (i,j) element of $M(R)$ is 1 if either $i \neq j$ and the subrotation system of $R(G)$ consisting of T and the edges e_i and e_j is non-planar (here we assign all edges in the subrotation system to be of type-0, regardless of the original edge type assignment in $R(G)$), or $i = j$ and the edge e_i is of type-1. Mohar's main theorem is as follows:

Theorem 1.1 (Mohar) *Let G, $R(G)$ and $M(R)$ be defined as above, then:*

(1). If $R(G)$ corresponds to an orientable imbedding, then the genus of $R(G)$ is equal to $\frac{1}{2}rank(M(R))$;

(2). If $R(G)$ corresponds to a nonorientable imbedding, then the crosscap number of $R(G)$ is equal to $rank(M(R))$.

Gross has interpreted the distribution of orientable imbeddings of cobblestone paths in terms of the overlap matrix [Gr]. In this paper, we show

some interesting properties of overlap matriecs. Using these properties, we give the distributions of nonorientable imbeddings of several interesting infinite families of graphs, including cobblestone paths and closed-end ladders for which the distributions of orientable imbeddings are known [FGS].

Our terminology is compatible with that of Gross and Tucker [GT], and of White [Wh].

Given a subset $S(G)$ of the set $R(G)$ of all rotation systems of a graph G. Suppose there are a_0 rotation systems in $S(G)$ corresponding to planar imbeddings of G, a_i rotation systems in $S(G)$ corresponding to orientable imbeddings of genus i of G, and b_j rotation systems in $S(G)$ corresponding to nonorientable imbeddings of crosscap number j of G. By two well-known theorems in topological graph theory, there are two integers I and J such that $a_i = 0$ and $b_j = 0$ for all $i > I$ and $j > J$. We define:

$$I_o(S(G), x) = \sum_{i=0}^{I} a_i x^i \qquad I_n(S(G), y) = \sum_{j=1}^{J} b_j y^j$$

and call $I_o(S(G), x)$ the "*genus distribution polynomial of G with respect to S(G)*", and $I_n(S(G), y)$ the "*crosscap number distribution polynomial of G with respect to S(G)*". If $S(G) = R(G)$ then we write $I_o(R(G), x)$ as $I_o(G, x)$ and call it "*the genus distribution polynomial of G*". Similarly we define "*the crosscap number distribution polynomial of G*" $I_n(G, y)$.

2 Some Properties of Overlap Matrix

Let M_n^O be an $n \times n$ symmetric matrix over $GF(2)$ of the following form:

$$M_n^O = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & \mathbf{O} & \\ & 1 & 0 & 1 & & & \\ & & & & \ddots & & \\ & \mathbf{O} & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}$$

That is, M_n^O is a tridiagonal matrix such that the diagonal elements are all 0, all elements in the two semidiagonals are 1, and all other elements are 0.

Lemma 2.1

$$rank(M_n^O) = \left\{ \begin{array}{ll} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{array} \right\} = 2\lfloor n/2 \rfloor$$

PROOF.

Straightforward induction. □

Now let $X = \{x_1, x_2, \cdots, x_n\} \in (GF(2))^n$. We define

$$M_n^X = \begin{pmatrix} x_1 & 1 & & & & & \\ 1 & x_2 & 1 & & & \mathbf{O} & \\ & 1 & x_3 & 1 & & & \\ & & & & \ddots & & \\ & \mathbf{O} & & & 1 & x_{n-1} & 1 \\ & & & & & 1 & x_n \end{pmatrix}$$

Then what is the rank of M_n^X? In fact, we are more interested in the distribution of ranks of M_n^X when X varies in the region $(GF(2))^n$.

Theorem 2.2 *Let R_n^k be the number of matrices M_n^X with rank k, where $X \in (GF(2))^n$. Then we have for $n > 0$:*

(1).

$$R_n^0 = R_n^1 = \cdots = R_n^{n-2} = 0$$

(2).

$$R_n^{n-1} = \left\{ \begin{array}{ll} (2^n+1)/3 & \text{if } n \text{ is odd} \\ (2^n-1)/3 & \text{if } n \text{ is even} \end{array} \right.$$

(3).

$$R_n^n = \left\{ \begin{array}{ll} \frac{2}{3}(2^n+1)-1 & \text{if } n \text{ is odd} \\ \frac{2}{3}(2^n-1)+1 & \text{if } n \text{ is even} \end{array} \right.$$

If we define a function round(r) to be the closest integer to the real number r, then we can write in a netter way [1]*:*

$$R_n^{n-1} = round(2^n/3) \qquad R_n^n = round(2^{n+1}/3)$$

[1] For people who do not like this notation, we point out that the *round* function can be expressed by a more common-uesd function: $round(r) = \lfloor r + 0.5 \rfloor$

PROOF.

Again we use induction on n.

It is a routine to check that the theorem is true for the cases $n = 1$ and $n = 2$. Now we suppose that the theorem is true for all $n < k$, where $k \geq 3$.

Suppose $n = 2m$. The matrix M_n^X can be written as

$$M_n^X = \begin{pmatrix} x_1 & 1 & & \mathbf{O} \\ 1 & x_2 & 1 & \\ & 1 & & \\ \mathbf{O} & & & M_{n-2}^{X_{n-2}} \end{pmatrix}$$

Where $X = \{x_1, x_2, \cdots, x_n\} \in (GF(2))^n$, $X_{n-2} = \{x_3, x_4, \cdots, x_n\} \in (GF(2))^{n-2}$, and $M_{n-2}^{X_{n-2}}$ is the corresponding $(n-2) \times (n-2)$ lower-right corner submatrix of M_n^X.

There are two cases:

(1). $x_1 = 1$. Then performing the standard matrix operations (see, for example, [BW]) on the first two rows and the first two columns of M_n^X, we can convert M_n^X into the following matrix without changing the rank of the matrix:

$$M_n^{(1)} = \begin{pmatrix} 1 & 0 & & \mathbf{O} \\ 0 & 1+x_2 & 1 & \\ & 1 & & \\ \mathbf{O} & & & M_{n-2}^{X_{n-2}} \end{pmatrix}$$

When $(x_2, x_3, \cdots, x_n)$ varies through $(GF(2))^{n-1}$, $(1 + x_2, x_3, \cdots, x_n)$ also varies through $(GF(2))^{n-1}$. Now in the 2^{n-1} possible lower-right corner $(n-1) \times (n-1)$ submatrices of $M_n^{(1)}$, by the inductive hypothesis, there are $(2^{n-1}+1)/3$ of them of rank $n-2$, and $\frac{2}{3}(2^{n-1}+1) - 1$ of them of rank $n-1$ (note that $n-1$ is odd). Since the first column of $M_n^{(1)}$ is linearly independent of all other columns in the matrix, we conclude that there are $(2^{n-1}+1)/3$ of $M_n^{(1)}$'s, thus M_n^X's, which are of rank $n-1$, and $\frac{2}{3}(2^{n-1}+1)-1$ of M_n^X's which are of rank n, if we restrict that $x_1 = 1$.

(2). $x_1 = 0$. Similarly, we can convert M_n^X into the following matrix without changing the rank of the matrix:

$$M_n^{(2)} = \begin{pmatrix} 0 & 1 & & \mathbf{O} \\ 1 & 0 & 0 & \\ & 0 & & \\ \mathbf{O} & & & M_{n-2}^{X_{n-2}} \end{pmatrix}$$

By the inductive hypothesis, there are $(2^{n-2}-1)/3$ of $M_{n-2}^{X_{n-2}}$'s of rank $n-3$, and $\frac{2}{3}(2^{n-2}-1)+1$ of $M_{n-2}^{X_{n-2}}$'s of rank $n-2$. Now since the first two columns of $M_n^{(2)}$ are linearly independent of all other columns in the matrix, we conclude that there are $(2^{n-2}-1)/3$ of $M_n^{(2)}$'s which are of rank $n-1$, and $\frac{2}{3}(2^{n-2}-1)+1$ of $M_n^{(2)}$'s which are of rank n. Finally, note that each $M_n^{(2)}$ corresponds to two different M_n^X's by setting either $x_2=1$ or $x_2=0$. We conclude that there are $2(2^{n-2}-1)/3$ of M_n^X's which are of rank $n-1$, and $\frac{4}{3}(2^{n-2}-1)+2$ of M_n^X's which are of rank n if we restrict that $x_1=0$.

Combining these two cases, we conclude:

$$R_n^{n-1} = (2^{n-1}+1)/3 + 2(2^{n-2}-1)/3 = (2^n-1)/3$$

$$R_n^n = \frac{2}{3}(2^{n-1}+1) - 1 + \frac{4}{3}(2^{n-2}-1) + 2 = \frac{2}{3}(2^n-1)+1$$

This completes the proof for the case $n=2m$.

For the case of $n=2m+1$, we can similarly get:

$$R_n^{n-1} = (2^{n-1}-1)/3 + 2(2^{n-2}+1)/3 = (2^n+1)/3$$

$$R_n^n = \frac{2}{3}(2^{n-1}-1) + 1 + \frac{4}{3}(2^{n-2}+1) - 2 = \frac{2}{3}(2^n+1) - 1$$

□

Let $X=\{x_1,x_2,\cdots,x_n\}\in(GF(2))^n$ and let $Y=\{y_1,y_2,\cdots,y_{n-1}\}\in(GF(2))^{n-1}$. Define

$$M_n^{X,Y} = \begin{pmatrix} x_1 & y_1 & & & & & \\ y_1 & x_2 & y_2 & & & \mathbf{O} & \\ & y_2 & x_3 & y_3 & & & \\ & & & & \ddots & & \\ & \mathbf{O} & & & y_{n-2} & x_{n-1} & y_{n-1} \\ & & & & & y_{n-1} & x_n \end{pmatrix}$$

Furthermore, let $S_n=\{M_n^{X,Y} \mid X\in(GF(2))^n \text{ and } Y\in(GF(2))^{n-1}\}$. We are interested in the distribution of ranks of matrices in S_n.

Let $i_1,i_2,\cdots,i_r$ be r positive integers such that $i_1+i_2+\cdots+i_r=n$. Suppose we choose a particular $Y_1\in(GF(2))^{n-1}$ such that $y_{i_1}=y_{i_1+i_2}=$

$\cdots = y_{i_1+i_2+\cdots+i_{r-1}} = 0$, and all other y_i's to be 1. For this particular Y_1, we have

$$M_n^{X,Y_1} = \begin{pmatrix} M_1 & & & O \\ & M_2 & & \\ & & \ddots & \\ O & & & M_r \end{pmatrix}$$

where each M_h is of the form $M_{i_h}^{X_h}$, with $X_h \in (GF(2))^{i_h}$. Note that each of these submatrices is independent of all others. Let X vary through $(GF(2))^n$. then each X_h varies through $(GF(2))^{i_h}$. Therefore, the distribution of ranks of M_n^{X,Y_1} when X varies through $(GF(2))^n$ is the convolution of the distributions of ranks of these $M_{i_h}^{X_h}$'s when each X_h varies through $(GF(2))^{i_h}$. If we define, similarly as for the distributions of graph imbeddings, the distribution polynomial $D(S, y)$ of ranks of a set S of matrices, i.e., we say the distribution polynomial of a set S of matrices is $D(S, y) = \sum_{i=0}^{n} c_i y^i$ if there are precisely c_i matrices in S of rank i for each i, then the above discussion gives us the following relations:

$$D(S(n.Y_1), y) = \prod_{h=1}^{r} D(S(i_h), y)$$

where $S(n, Y_1) = \{M_n^{X,Y_1} \mid X \in (GF(2))^n\}$ and $S(i_h) = \{M_{i_h}^{X_h} \mid X_h \in (GF(2))^{i_h}\}$. By theorem 2.2 we know that

$$D(S(i_h), y) = \text{round}(2^{i_h}/3)y^{i_h-1} + \text{round}(2^{i_h+1}/3)y^{i_h}$$

thus

$$D(S(n, Y_1), y) = \prod_{h=1}^{r} (\text{round}(2^{i_h}/3)y^{i_h-1} + \text{round}(2^{i_h+1}/3)y^{i_h})$$

Since each selection of the set of positive integers $i_1, i_2, \cdots, i_r$ satisfying $i_1 + i_2 + \cdots + i_r = n$ gives a unique $Y_1 \in (GF(2))^{n-1}$, thus a unique decomposition of M_n^{X,Y_1} into the above form, and vice versa, we get finally the distribution polynomial of the set S_n:

Theorem 2.3

$$D(S_n, y) = \sum_{i_1,\cdots,i_r>0}^{i_1+i_2+\cdots+i_r=n} \prod_{h=1}^{r} (round(2^{i_h}/3)y^{i_h-1} + round(2^{i_h+1}/3)y^{i_h})$$

3 Distributions of Nonorientable Imbeddings of Graphs

Suppose that every edge of the n-vertex path is doubled, and that a self-adjacency is then added at each end. The resulting graph is called a *cobblestone path of length* n, written as J_n. The following picture is a cobblestone path of length 6.

Fix a spanning tree T of J_{n-1} (which must consist of edges which connect each pair of adjacent vertices). In the above figure the tree edges are shown as thicker lines. Each co-tree edge determines a unique cycle in the graph and two co-tree edges can overlap only if their corresponding cycle have a vertex in common. This implies, in the case of a cobblestone path, that only adjacent co-tree edges can overlap. Given a rotation system $R(J_{n-1})$ of J_{n-1}, we construct the corresponding overlap matrix M_n for $R(J_{n-1})$. M_n is an $n \times n$ matrix. Organizing the rows and columns of the overlap matrix M_n so that consecutive rows (columns) correspond to adjacent co-tree edges. Then M_n must be of the following form:

$$M_n = M_n^{X,Y} = \begin{pmatrix} x_1 & y_1 & & & & & \\ y_1 & x_2 & y_2 & & & \mathbf{O} & \\ & y_2 & x_3 & y_3 & & & \\ & & & & \ddots & & \\ & \mathbf{O} & & & y_{n-2} & x_{n-1} & y_{n-1} \\ & & & & & y_{n-1} & x_n \end{pmatrix}$$

where $X = \{x_1, x_2, \cdots, x_n\} \in (GF(2))^n$ and $Y = \{y_1, y_2, \cdots, y_{n-1}\} \in (GF(2))^{n-1}$. Note that each variable y_i corresponds to a unique vertex of the cobblestone path J_{n-1} and has value 1 if and only if the two co-tree

edges incident to that vertex overlap, and each variable x_i corresponds to a unique co-tree edge of J_{n-1} and has value 1 if and only if the edge is "twisted". Mohar's theorem says that if $R(J_{n-1})$ corresponds to a nonorientable imbedding of J_{n-1}, then the rank of M_n is equal to the crosscap number of $R(J_{n-1})$. Now if we fix a pure rotation system $R_o(J_{n-1})$ of J_{n-1} (i.e., fix the cyclic ordering of edges adjacent to each vertex of J_{n-1}), and consider all possible twistings of co-tree edges. This corresponds to fixing $Y = \{y_1, y_2, \cdots, y_{n-1}\}$ in the above matrix $M_n = M_n^{X,Y}$ and letting $X = \{x_1, x_2, \cdots, x_n\}$ vary through $(GF(2))^n$. Let $S(R_o)$ be the set of the 2^n these kind of rotation systems of J_{n-1} and let $S(M_n)$ be the set of the corresponding 2^n matrices. By theorem 2.3, the distribution polynomial of the set $S(M_n)$ is

$$D(S(M_n), y) = \prod_{h=1}^{r} (\text{round}(2^{i_h}/3)y^{i_h-1} + \text{round}(2^{i_h+1}/3)y^{i_h})$$

where the integers $i_1, i_2, \cdots, i_r$ correspond to the zero elements in $Y = \{y_1, y_2, \cdots, y_{n-1}\}$, as we have described in the previous section. There is only one matrix in $S(M_n)$ corresponding to an orientable imbedding of J_{n-1} (the one with all $x_i = 0$). By lemma 2.1, this matrix has rank $c(i_1, \cdots, i_r) = \sum_{h=1}^{r} 2\lfloor i_h/2 \rfloor$. Therefore, the crosscap number distribution polynomial of J_{n-1} with respect to $S(R_o)$ is

$$I_n(S(R_o), y) = \prod_{h=1}^{r} (\text{round}(2^{i_h}/3)y^{i_h-1} + \text{round}(2^{i_h+1}/3)y^{i_h}) - y^{c(i_1,\cdots,i_r)}$$

Each vertex of J_{n-1} has degree 4 and, thus, there are 6 possible rotations at each vertex. Of the six rotations, exactly two require the incident co-tree edges to cross each other. It follows that in $Y = \{y_1, y_2, \cdots, y_{n-1}\}$, there are two ways to set each y_i to 1 and four ways to set each y_i to 0. Therefore, let $S(i_1, \cdots, i_r)$ be the set of all rotation systems of J_{n-1} whose corresponding overlap matrix is in $S(M_n)$, then the crosscap number distribution polynomial of J_{n-1} with respect to $S(i_1, \cdots, i_r)$ is

$$R(S(i_1, \cdots, i_r), y) = 4^{r-1}2^{n-1-(r-1)} I_n(S(R_o), y)$$

$$= 2^{n+r-2}(\prod_{h=1}^{r} (\text{round}(2^{i_h}/3)y^{i_h-1} + \text{round}(2^{i_h+1}/3)y^{i_h}) - y^{c(i_1,\cdots,i_r)})$$

Now consider all possible pure rotation systems of J_{n-1}, which corresponds to all possible choices of $Y = \{y_1, y_2, \cdots, y_{n-1}\}$ in $(GF(2))^{n-1}$. Summarizing all these together, we get the crosscap number distribution polynomial of J_{n-1} as follows:

$$I_n(J_{n-1}, y) =$$

$$= \sum_{i_1,\cdots,i_r>0}^{i_1+\cdots+i_r=n} 2^{n+r-2} \prod_{h=1}^{r} (\text{round}(2^{i_h}/3)y^{i_h-1} + \text{round}(2^{i_h+1}/3)y^{i_h})$$

$$- \sum_{i_1,\cdots,i_r>0}^{i_1+\cdots+i_r=n} 2^{n+r-2} y^{c(i_1,\cdots,i_r)}$$

The last term $\sum_{i_1,\cdots,i_r>0}^{i_1+\cdots+i_r=n} 2^{n+r-2} y^{c(i_1,\cdots,i_r)}$ corresponds to the orientable imbeddings of J_{n-1}, which is known [FGS]. Also note that the rank of an overlap matrix is 2 times the genus of the corresponding orientable imbedding. Therefore, if we let the genus distribution polynomial of J_{n-1} be $I_o(J_{n-1}, x)$, then the last term can be expressed by $I_o(J_{n-1}, y^2)$. Finally we get:

$$I_n(J_{n-1}, y) + I_o(J_{n-1}, y^2) =$$

$$= \sum_{i_1,\cdots,i_r>0}^{i_1+\cdots+i_r=n} 2^{n+r-2} \prod_{h=1}^{r} (\text{round}(2^{i_h}/3)y^{i_h-1} + \text{round}(2^{i_h+1}/3)y^{i_h})$$

Another interesting infinite family of graphs is closed-end ladders. A *n-rung closed-end ladder* L_n can be obtained by taking the graphical Cartesian product of a n-vertex path P_n with the complete graph K_2, and then doubling both its end edges. The following picture gives a 4-rung closed-end ladder:

Let us consider the crosscap number distribution of an $(n-1)$-rung closed-end ladder. If we select the spanning tree as shown by the thicker lines in the above picture, and notice that now there are 2^{n-1} different rotation systems for the selected spanning tree, but once a rotation system for the spanning tree is fixed, there is a unique way to set each y_i to 1 and a unique way to set each y_i to 0. A completely similar analysis as that we have given for cobblestone paths gives us the crosscap number distribution polynomial of a $(n-1)$-rung closed-end ladder L_{n-1} as follows:

$$I_n(L_{n-1}, y) + I_o(L_{n-1}, y^2) =$$

$$= 2^{n-1} \sum_{i_1,\cdots,i_r>0}^{i_1+\cdots+i_r=n} \prod_{h=1}^{r} (\text{round}(2^{i_h}/3) y^{i_h-1} + \text{round}(2^{i_h+1}/3) y^{i_h})$$

where $I_o(L_{n-1}, x)$ is the genus distribution polynomial of the closed-end ladder L_{n-1} which has been know [FGS].

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