

**A Linear-Time Algorithm
for Concave One-Dimensional Dynamic Programming**

Zvi Galil
Kunsoo Park

CUCS-469-89

A Linear-Time Algorithm for Concave One-Dimensional Dynamic Programming*

Zvi Galil^{1,2} and Kunsoo Park¹

¹ Department of Computer Science, Columbia University, New York, NY 10027

² Department of Computer Science, Tel-Aviv University, Tel-Aviv, Israel

Keywords: Dynamic programming, quadrangle inequality, total monotonicity

The one-dimensional dynamic programming problem is defined as follows: given a real-valued function $w(i, j)$ for integers $0 \leq i \leq j \leq n$ and $E[0]$, compute

$$E[j] = \min_{0 \leq i < j} \{D[i] + w(i, j)\}, \quad \text{for } 1 \leq j \leq n,$$

where $D[i]$ is computed from $E[i]$ in constant time. The least weight subsequence problem [4] is a special case of the problem where $D[i] = E[i]$. The modified edit distance problem [3], which arises in molecular biology, geology, and speech recognition, can be decomposed into $2n$ copies of the problem.

Let A be an $n \times m$ matrix. $A[i, j]$ denotes the element in the i th row and the j th column. $A[i : i', j : j']$ denotes the submatrix of A that is the intersection of rows $i, i+1, \dots, i'$ and columns $j, j+1, \dots, j'$. We say that the cost function w is *concave* if it satisfies the quadrangle inequality [7]

$$w(a, c) + w(b, d) \leq w(b, c) + w(a, d), \quad \text{for } a \leq b \leq c \leq d.$$

In the concave one-dimensional dynamic programming problem w is concave as defined above. A condition closely related to the quadrangle inequality was introduced by Aggarwal et al. [1]. An $n \times m$ matrix A is *totally monotone* if for all $a < b$ and $c < d$,

$$A[a, c] > A[b, c] \implies A[a, d] > A[b, d].$$

Let $r(j)$ be the smallest row index such that $A[r(j), j]$ is the minimum value in column j . Then total monotonicity implies

$$r(1) \leq r(2) \leq \dots \leq r(m). \quad (1)$$

That is, the minimum row indices are nondecreasing. We say that an element $A[i, j]$ is *dead* if $i \neq r(j)$. A submatrix of A is dead if all of its elements are dead. Note that for $a \leq b \leq c \leq d$, the quadrangle inequality implies total monotonicity, but the converse is not true. Aggarwal et al. [1] show that the row maxima of a totally monotone $n \times m$ matrix A can be found in $O(n + m)$ time if $A[i, j]$ for any i, j can be computed in constant time. Their algorithm is easily adapted to find the column minima. We will refer to their algorithm as the SMAWK algorithm.

Let $B[i, j] = D[i] + w(i, j)$ for $0 \leq i < j \leq n$. We say that $B[i, j]$ is *available* if $D[i]$ is known and therefore $B[i, j]$ can be computed in constant time. Then the problem is to find the column minima in the upper triangular matrix B with the restriction that $B[i, j]$ is available only after the column minima for columns $1, 2, \dots, i$ have been found. It is easy to see that when w satisfies the quadrangle inequality, B also satisfies the quadrangle inequality. For the concave problem Hirschberg and Larmore [4] and later Galil and Giancarlo [3] gave $O(n \log n)$ algorithms using queues. Wilber [6] proposed an $O(n)$ time algorithm when $D[i] = E[i]$. However, his algorithm does not work if the availability of matrix B must be obeyed, which happens when many copies

* Work supported in part by NSF Grants CCR-86-05353 and CCR-88-14977

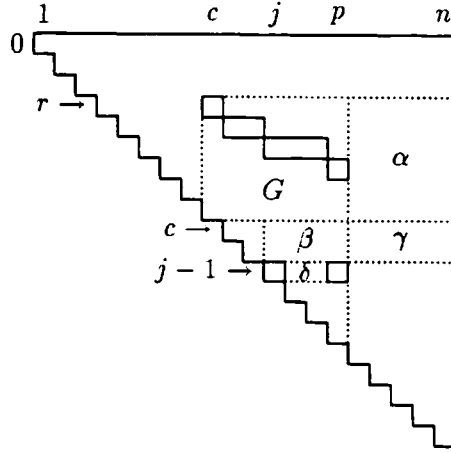


Figure 1. Matrix B at a typical iteration

of the problem proceed simultaneously (i.e., the computation is interleaved among many copies) as in the modified edit distance problem [3] and the mixed convex and concave cost problem [2]. Eppstein [2] extended Wilber's algorithm for interleaved computation. Our algorithm is more general than Eppstein's; it works for any totally monotone matrix B (we use only relation (1)), whereas Eppstein's algorithm works only when $B[i, j] = D[i] + w(i, j)$. Our algorithm is also simpler than both Wilber's and Eppstein's. Recently, Larmore and Schieber [5] reported another linear-time algorithm, which is quite different from ours.

The algorithm consists of a sequence of iterations. Figure 1 shows a typical iteration. We use $N[j]$, $1 \leq j \leq n$, to store interim column minima before row r ; $N[j] = B[i, j]$ for some $i < r$ (the usage will be clear shortly). At the beginning of each iteration the following invariants hold:

- (a) $0 \leq r$ and $r < c$.
- (b) $E[j]$ for all $1 \leq j < c$ have been found.
- (c) $E[j]$ for $j \geq c$ is $\min(N[j], \min_{i \geq r} B[i, j])$.

Invariant (b) means that $D[i]$ for all $0 \leq i < c$ are known, and therefore $B[i, j]$ for $0 \leq i < c$ and $c \leq j \leq n$ is available. Initially, $r = 0$, $c = 1$, and all $N[j]$ are $+\infty$.

Let $p = \min(2c - r, n)$, and let G be the union of $N[c : p]$ and $B[r : c - 1, c : p]$, $N[c : p]$ as its first row and $B[r : c - 1, c : p]$ as the other rows. G is a $(c - r + 1) \times (c - r + 1)$ matrix unless $2c - r > n$. Let $F[j]$, $c \leq j \leq p$, denote the column minima of G . Since matrix G is totally monotone, we use the SMAWK algorithm to find the column minima of G . Once $F[c : p]$ are found, we compute $E[j]$ for $j = c, c + 1, \dots$ as follows. Obviously, $E[c] = F[c]$. For $c + 1 \leq j \leq p$, assume inductively that $B[c : j - 2, j : p]$ (β in Figure 1) is dead and $B[j - 1, j : n]$ is available. It is trivially true when $j = c + 1$. By the assumption $E[j] = \min(F[j], B[j - 1, j])$.

- (1) If $B[j - 1, j] < F[j]$, then $E[j] = B[j - 1, j]$, and by relation (1) $B[r : j - 2, j : n]$ (α, β, γ , and the part of G above β in Figure 1) and $N[j : n]$ are dead. We start a new iteration with $c = j + 1$ and $r = j - 1$.
- (2) If $F[j] \leq B[j - 1, j]$, then $E[j] = F[j]$. We compare $B[j - 1, p]$ with $F[p]$.
 - (2.1) If $B[j - 1, p] < F[p]$, $B[r : j - 2, p + 1 : n]$ (α and γ in Figure 1) is dead by relation (1). $B[c : j - 2, j : p]$ (β in Figure 1) is dead by the assumption. Thus only $F[j + 1 : p]$ among $B[0 : j - 2, j + 1 : n]$ may become column minima in the future computation. We store $F[j + 1 : p]$ in $N[j + 1 : p]$ and start a new iteration with $c = j + 1$ and $r = j - 1$.
 - (2.2) If $F[p] \leq B[j - 1, p]$, $B[j - 1, j : p]$ (δ in Figure 1) is dead by relation (1) in submatrix $B[r : j - 1, j : p]$ (β, δ , and the part of G above β). Since $B[j, j + 1 : n]$ is available from

```

procedure concave 1D
   $c \leftarrow 1$ ;
   $r \leftarrow 0$ ;
   $N[1:n] \leftarrow +\infty$ ;
  while  $c \leq n$  do
     $p \leftarrow \min(2c - r, n)$ ;
    use SMAWK to find column minima  $F[c:p]$  of  $G$ ;
     $E[c] \leftarrow F[c]$ ;
    for  $j \leftarrow c + 1$  to  $p$  do
      if  $B[j - 1, j] < F[j]$  then
         $E[j] \leftarrow B[j - 1, j]$ ;
        break
      else
         $E[j] \leftarrow F[j]$ ;
        if  $B[j - 1, p] < F[p]$  then
           $N[j + 1:p] \leftarrow F[j + 1:p]$ ;
          break
        end if
      end if
    end for
    if  $j \leq p$  then
       $c \leftarrow j + 1$ ;
       $r \leftarrow j - 1$ 
    else
       $c \leftarrow p + 1$ ;
       $r \leftarrow \max(r, \text{row of } F[p])$ 
    end if
  end while
end

```

Figure 2. The algorithm for concave 1D dynamic programming

$E[j]$, the assumption holds at $j + 1$. We go on to column $j + 1$.

If case (2.2) is repeated until $j = p$, we have found $E[j]$ for $c \leq j \leq p$. We start a new iteration with $c = p + 1$. If the row of $F[p]$ is greater than r , it becomes the new r (it may be smaller than r if it is the row of $N[p]$). Note that the three invariants hold at the beginning of new iterations. Figure 2 shows the algorithm, where the **break** statement causes the innermost enclosing loop to be exited immediately.

Each iteration takes time $O(c - r)$. If either case (1) or case (2.1) happens, we charge the time to rows $r, \dots, c - 1$ because r is increased by $(j - 1) - r \geq c - r$. If case (2.2) is repeated until $j = p$, there are two cases. If $p < n$, we charge the time to columns c, \dots, p because c is increased by $(p + 1) - c \geq c - r + 1$. If $p = n$, we have finished the whole computation, and rows $r, \dots, c - 1 (< n)$ have not been charged yet; we charge the time to the rows. Since c and r never decrease, only constant time is charged to each row or column. Thus the total time of the algorithm is linear in n .

References

- [1] Aggarwal, A., Klawe, M. M., Moran, S., Shor, P., and Wilber, R. Geometric applications of a matrix-searching algorithm. *Algorithmica* **2** (1987), 195–208.
- [2] Eppstein, D. Sequence comparison with mixed convex and concave costs. *J. Algorithms* to appear.
- [3] Galil, Z., and Giancarlo, R. Speeding up dynamic programming with applications to molecular biology. *Theoretical Computer Science* **64** (1989), 107–118.
- [4] Hirschberg, D. S., and Larmore, L. L. The least weight subsequence problem. *SIAM J. Comput.* **16**, 4 (1987), 628–638.
- [5] Larmore, L. L., and Schieber, B. On-line dynamic programming with applications to the prediction of RNA secondary structure. *to be presented at the First Annual ACM-SIAM Symposium on Discrete Algorithms*.
- [6] Wilber, R. The concave least-weight subsequence problem revisited. *J. Algorithms* **9** (1988), 418–425.
- [7] Yao, F. F. Speed-up in dynamic programming. *SIAM J. Alg. Disc. Meth.* **3** (1982), 532–540.