HIERARCHY FOR IMBEDDING-DISTRIBUTION

INVARIANTS OF A GRAPH

Jonathan L. Gross and Merrick Furst

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by Jonathan L. Gross[1] and Merrick Furst[2]

Most existing papers about graph imbeddings are concerned with the determination of minimum genus, and various others have been devoted to maximum genus or to highly symmetric imbeddings of special graphs. An entirely different viewpoint is now presented, in which one seeks distributional information about the huge family of all cellular imbeddings of a graph into all closed surfaces, instead of focusing on just one imbedding or on the existence of imbeddings into just one surface. The distribution of imbeddings admits a hierarchically ordered class of computable invariants, each of which partitions the set of all graphs into much finer subcategories than the subcategories corresponding to minimum genus or to any other single imbedding surface. Quite low in this hierarchy are invariants such as the average genus, taken over all cellular imbeddings, and the average region size, where "region size" means the number of edge traversals required to complete a tour of a region boundary. Further up in the hierarchy is the multiset of duals of a graph. At an intermediate level are the "imbedding polynomials". The hierarchy is explored, and several specific calculations of the values of some of the invariants are provided. The main results are concerned with the amount of work needed to derive one invariant from another, when possible, and with principles for computing the algebraic effect of adding an edge or of forming a composition of two graphs.

1 The genus distribution of a graph

Given any graph, we would like to know how many essentially different imbeddings it has in each type of surface. Before discussion of this problem, it will be helpful to review some of the standard objects and terminology of topological graph theory.

Throughout this paper, a *graph* may have multiple adjacencies or self-adjacencies. It is taken to be connected, unless one can readily infer otherwise from the immediate context.

The closed orientable surface of genus i is denoted S_i. By an *imbedding*, we mean a polygonal imbedding into a closed orientable surface. However, the methods and results of this paper can be broadened to encompass non-orientable imbeddings as well.

Two imbeddings are called *equivalent* if one can be continuously deformed onto the other. We regard equivalent imbeddings as the "same", so that when we say we are counting imbeddings, it means we are counting equivalence classes.

The *size of the region (or face)* of an imbedding means the number of edge-traversals required to complete a tour of the region boundary. If both orientations of the same edge are encountered on the tour, then both contribute to the size.

It is assumed that the reader is familiar with the basics of topological graph theory, at the level of White and Beineke [1978], with the possible exception of covering space constructions (i.e., voltage graphs and current graphs), or Chapter 1 of Gross and Tucker [1986]. The following exposition of the relationship between rotation systems and graph imbeddings increases the generality somewhat.

A *rotation* at a vertex is a cyclic permutation of the edges incident on it. In the case of a self-adjacency, each of the ends appears separately in the rotation. It follows that a vertex with valence d admits (d - 1)! different rotations.

It is customary to specify a rotation for a vertex v in the format

$$v. \quad e_1 \quad e_2 \quad \cdots \quad e_d$$

and to place sign marks ("+" or "-") on edges when they are needed to specify edge-direction and thereby

to avoid ambiguity. A list of rotations, one for each vertex, is called a *rotation system* for the graph. Similarly, various abstractions of such a list are also called a *rotation system*. If a graph has vertices $v_1,...,v_n$ of respective valences $d_1,...,d_n$, then the total number of rotations systems is

$$\prod_{j=1}^{n} (d_j-1)!$$

There is a bijective correspondence between the family of imbeddings of a graph and the set of rotation systems. Edmonds [1960] was first to call explicit attention to this correspondence, which seems to have been understood earlier by Heffter [1891]. Details of the correspondence were first published by Youngs [1963], for the simplicial case. A full generalization to allow for multiple-adjacencies and self-adjacencies was developed by Gross and Alpert [1974].

Here is the underlying idea of the correspondence. Suppose that e is an edge from vertex u to vertex v. There is a permutation action on the set of oriented edges, which is induced by the rotation system. This action permutes e to the reverse of the oriented edge that follows e in the rotation at v. The edge-orbits of the action are taken to be the face boundaries of an imbedding.

From this point on, some of the definitions and notational devices are new.

For any connected graph G, the number of (polygonal) imbeddings of G in the surface S_j is denoted $g_j(G)$, or simply g_j, if there is no problem of ambiguity in regard to the graph.

The minimum value of j such that $g_j(G)$ is positive is called the *minimum genus* of G (or often, elsewhere, just the *genus* of G), and is denoted $\gamma_{min}(G)$. Similarly, the maximum value of j such that $g_j(G)$ is positive is called the *maximum genus* of G, and is denoted $\gamma_{max}(G)$.

The interval $[\gamma_{min}, \gamma_{max}]$ is called the *genus range*. We abuse the terminology slightly by saying that a surface is in the genus range, when we mean that its genus is there.

Theorem 1.1 (Duke [1966]) Let G be any connected graph. For every integer j in the genus range, the number $g_j(G)$ is positive.

Proof: Imagine a "path" from a minimum-genus rotation system to a maximum-genus rotation system, in which each "step" is a transposition of two adjacent edges in the rotation at some vertex.

Since each step changes the genus of the imbedding surface by a most one, it follows that every surface in the genus range admits at least one imbedding of the graph G. This theorem is sometimes called "Duke's interpolation principle".

<div align="right">QED</div>

By the *genus distribution* of G, we mean the sequence

$$g_0(G), \ g_1(G), \ g_2(G), \ ...$$

Sometimes we encode it as a *genus polynomial*

$$g[G](x) \ = \ g_0 \ + \ g_1 x \ + \ g_2 x^2 \ + \ ...$$

Since the genus range is finite, this is a finite polynomial. Aside from Theorem 1.1, there is very little general information known about the genus distribution. For instance, it is not known whether it is unimodal. Published discussion of the average genus γ_{avg} or the standard deviation γ_{sd} is rare. Stahl [1983] has studied the average genus of a class of graphs with the same number of edges, but not of individual graphs within the class.

Here are a few examples of a genus distribution for individual graphs. Routine computational details are omitted.

Example 1 Each of the four vertices of the complete graph K_4 has valence three. Therefore, the number of imbeddings is

$$((3 - 1)!)^4 \ = \ 16$$

Two of them are in the sphere S_0, and the other 14 are in the torus S_1. It follows that

$$g[K_4](x) \ = \ 2 \ + \ 14x$$

and that $\gamma_{avg}(K_4) = 7/8$.

Example 2 The complete bipartite graph $K_{3,3}$ has six vertices, each of valence three. Hence, there are

$$((3 - 1)!)^6 \ = \ 64$$

imbeddings. Of these, 40 are in S_1 and 24 are in S_2. Thus,

$$g[K_{3,3}](x) = 40x + 24x^2$$

The average genus is 11/8.

Example 3 The prism graph $K_3 \times K_2$ also has 64 imbeddings. However, its genus polynomial is

$$2 + 38x + 24x^2$$

and its average genus is 43/32.

In general, one might expect greater or much greater difficulty in calculating the entire genus distribution of a graph than just its minimum genus. For any particular graph or class of graphs, such as the complete graphs, one surely expects that the determination of a formula for the entire distribution would be more difficult, perhaps by far, than the derivation of a formula for minimum genus.

On the other hand, it might be possible to make careful estimates of the average genus and other properties of the genus distribution from a tractably sized random sample of imbeddings. At the present time, it appears that very little is known about the genus distribution as a graph invariant.

Formula derivations for infinite classes of graphs are among the lines of research that might be pursued in a general program to study genus distribution. Furst, Gross, and Statman [1985] have initiated this effort with a calculation for two infinite classes in which the maximum genus increases indefinitely with the size of the graphs.

There are two classes of graphs for which the genus distribution calculation is trivial. Obviously, the sphere is the only surface that admits cellular imbeddings of a tree.

Furthermore, let us call a connected graph whose number of edges equals its number of vertices a *bushy cycle*. Like trees, bushy cycles have no non-spherical orientable imbeddings. As a generalization of a bushy cycle, let G be any graph such that at most one distinct unoriented (simple) cycle passes through any vertex, so that G looks like some bushy cycles with a tree of connecting bars among them.

We will call such a graph a *tree of bushy cycles*.

In order to establish that a tree of bushy cycles has no imbedding surfaces except the sphere, it is helpful to use the work of Xuong [1979]. We begin by defining the *deficiency* of a spanning tree T in a graph G to be the number of components of the edge-complement G - T that have an even number of edges. The minimum deficiency for any spanning tree in G is called the *span deficiency* of G, and is denoted $\xi(G)$. Xuong proved that every connected graph G satisfies the equation.

$$\gamma_{\max}(G) \;=\; 1/2 \,(\beta(G) - \xi(G))$$

where $\beta(G)$ is the *cycle rank*, #edges - #vertices + 1. The right-hand side of that equation is called "Xuong's formula".

Every spanning tree in a tree of bushy cycles includes all the "connecting bars", and leaves exactly one edge remaining in each of the cycles. It follows that the span deficiency equals the cycle rank. By Xuong's formula, $\gamma_{\max} = 0$, which implies that the only possible imbedding surface is the sphere.

Theorem 1.2   (Nordhaus, Ringeisen, Stewart, and White [1972]) Let G be a graph that is neither a tree nor a tree of bushy cycles. Then the genus range of G contains more than one integer.

Proof: This theorem simply asserts part of Theorem 4 of the cited paper.

QED

2 Region sizes and the imbedding polynomials

The genus distribution is a topological invariant of a graph. That is, if two graphs are homeomorphic, then they have the same genus distribution. Our attention now turns to some invariants based on region size, which are obviously invariants of the combinatorial isomorphism class, but not necessarily topological invariants, unless one restricts consideration to graphs with no vertices of valence two.

For any connected graph G, let $f_j(G)$ denote the number of j-sided regions of G, taken over all orientable imbeddings. If there is no confusion as to which graph is the object, we may write f_j. Like the genus distribution, the region-size distribution may be encoded as a polynomial

$$s[G](y) \; = \; f_0 \; + \; f_1 y \; + \; f_2 y^2 \; + \; ...$$

This polynomial has finitely many non-zero coefficients, and it is called the *region-size polynomial* or the *face-size polynomial*.

Unlike the genus polynomial, the region-size polynomial has zero coefficients interspersed among the positive coefficients. For instance, a bipartite graph has no non-zero coefficients of odd index.

Example 1, revisited Both spherical imbeddings of the complete graph K_4 have four 3-sided faces and no others. Six of the toroidal imbeddings have one 4-sided face and one 8-sided face. The other eight toroidal imbeddings have one 3-sided face and one 9-sided face. Therefore,

$$s[K_4](y) \; = \; 16y^3 \; + \; 6y^4 \; + \; 6y^8 \; + \; 8y^9$$

It follows that the average region size is $16/3$.

It is often helpful to encode the combination of region sizes of an imbedding as a multivariate monomial, in the following manner. For each positive integer j, the exponent of the variable z_j equals the number of j-sided regions in the imbedding. The sum of these monomials, taken over all imbeddings, is called the *imbedding polynomial* for the graph G. It is denoted $i[G](z_j)$. In particular,

$$i[K_4](z_j) \; = \; 2z_3{}^4 \; + \; 6z_4 z_8 + 8z_3 z_9$$

Example 2, revisited By some routine calculations, whose details are omitted, we can determine

that

$$s[K_{3,3}](y) = 72y^4 + 12y^6 + 36y^{10} + 24y^{18}$$

that

$$i[K_{3,3}](z_j) = 4z_6{}^3 + 36z_4{}^2 z_{10} + 24z_{18}$$

and that the average region size is 8.

Example 3, revisited   Although the genus polynomial for the prism graph $K_3 \times K_2$ is numerically close to the genus polynomial for the complete bipartite graph $K_{3,3}$, their region-size polynomials and imbedding polynomials are quite dissimilar. In particular,

$$s[K_3 \times K_2](y) = 32y^3 + 28y^4 + 24y^5 + 2y^6 + 2y^8 + 14y^{10} + 20y^{11} + 2y^{12} + 24y^{18}$$

and

$$i[K_3 \times K_2](z_j) = 2z_3{}^2 z_4{}^3 + 2z_3{}^2 z_{12} + 20z_3 z_4 z_{11} + 4z_3 z_5 z_{10} + 2z_4 z_6 z_8 + 10z_5{}^2 z_8 + 24z_{18}$$

Moreover, the average region size is $288/37$.

So far, we have considered eight different invariants of the family of imbeddings of a graph. The following observations establish some of their interrelationships in the hierarchy. Figure 2.1 below provides a convenient summary.

Theorem 2.1 The region-size polynomial of a graph can be calculated from the imbedding polynomial, without recourse to any additional information whatsoever about the graph.

Proof: The number of edges of the graph can be recovered immediately from any term of the imbedding polynomial, and the possible region sizes range between the number one and twice the number of edges. Given an imbedding polynomial, one initializes every entry in an array of $2\#E$ counters to zero. Then one scans the imbedding polynomial term by term. For each term, one increments the appropriate entries in the array, corresponding to the subscripts enountered, the appropriate number of times, corresponding to the exponents of the respective variables.   QED

The time needed to calculate the region-sized polynomial depends mainly on the number of terms in

the imbedding polynomial. Since each term of the imbedding polynomial corresponds to a partition of 2#E, and since the number of partitions of n is known to grow as an exponential function of the argument n, there is a possibility that imbedding polynomials have excessively many terms. In Section 5 a number-theoretic argument is used to prove that the number of terms of the imbedding polynomial actually does grow exponentially with respect to the number of edges.
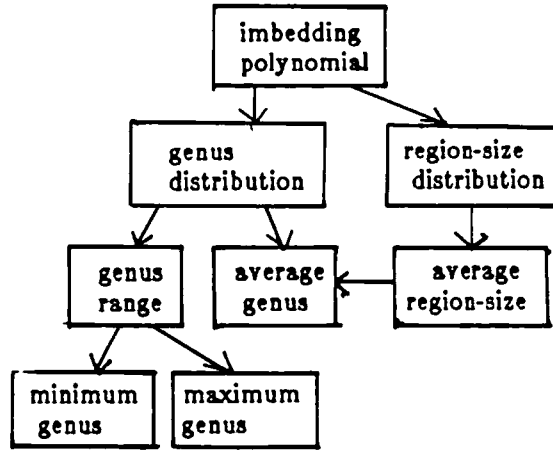

Figure 2.1 The hierarchical relationships among some of the invariants of the imbedding space.

Theorem 2.2 The genus polynomial of a graph can be calculated from the imbedding polynomial, provided that the number of vertices is known.

Proof: First, convert the multivariate imbedding polynomial $i[G](z_j)$ into a univariate polynomial $i[G](z)$ in z alone, by dropping all the subscripts. Then the coefficient of the term z^r is the number of imbeddings with r regions. For instance, in Example 3,

$$i[K_3 \times K_2](z) = 2z^5 + 38z^3 + 24z$$

The Euler equation

$$2 - 2\gamma = \#V - \#E + \#F$$

enables one to convert each term of the univariate polynomial $i[G](z)$ into a term of the genus polynomial for the graph G.

<div align="right">QED</div>

Yet another observation incorporated into Figure 2.1 is that the average genus and the average face-size are equivalent information, provided that the number of vertices is known. To establish this, we simply observe that it is a routine matter to derive an "average Euler equation"

$$2 - 2\gamma_{avg} = \#V - \#E + \#F_{avg}$$

and an "average dual" formulation

$$\#F_{avg} = 2\#E \ / \ (avg \ face\text{-}size)$$

of Euler's theorem about the sum of the valences of a graph.

which is already known from Example 1 to be the imbedding polynomial for K_4.

Next, let us suppose that one of the vertices u and v is of valence three in $K_4 - e$ and that the other is of valence two. (No matter which vertex of valence two is chosen and no matter which vertex of valence three, the total choice is equivalent to any other such choice, for the graph $K_4 - e$.) Then the augmented imbedding polynomial is

$$2z_3(u_{\bullet\bullet})z_3(uv_{\bullet})z_4(uv_{\bullet\bullet}) \; + \; 2z_{10}(uv_{\bullet\bullet}u_{\bullet}vu_{\bullet\bullet})$$

When the existing edge between u and v is doubled, the resulting graph has the imbedding polynomial

$$4z_2z_3^2z_4 \; + \; 4z_2z_{10} \; + \; 8z_3z_9 \; + \; 4z_4z_8 \; + \; 4z_5z_7$$

Finally, in the third way to select two vertices of $K_4 - e$, both of them have valence three. In this case the augmented imbedding polynomial is

$$2z_3(uv_{\bullet})z_3(uv_{\bullet})z_4(uv_{\bullet\bullet}) \; + \; 2z_{10}(u_{\bullet}v_{\bullet}uv_{\bullet}u_{\bullet}v)$$

It follows that doubling the edge between u and v yields a graph with the imbedding polynomial

$$6z_2z_3^2z_4 \; + \; 4z_2z_{10} \; + \; 16z_3z_9 \; + \; 4z_4z_8 \; + \; 6z_5z_7$$

Without elaborating in a full theoretical discourse, we would like to mention the possibility of augmenting an imbedding polynomial by additional information regarding the location of arbitrary subgraphs, not just vertex pairs.

3 The effect of adding an edge; augmented polynomials

We have now established the concepts of a genus distribution and of a distribution of imbeddings. We have also calculated them for a few simple examples. It is now time to lay the foundation for computing the distributions of larger graphs, by investigating the effect of adding a single edge. Furst, Gross, and Statman [1985] have already used the principle described in this section, in order to obtain genus polynomials for each of an infinite sequence of graphs, each obtained from its precedessor by the addition of one edge.

The key idea is that before we add an edge to a graph G, we want to *augment* the imbedding polynomial for G so as to mark the terms and variables corresponding to the prospective endpoints. The following example illustrates the marking process.

Example 4 The graph K_4 - e (i.e. the complement in K_4 of a one-edge subgraph) has the imbedding polynomial

$$i[K_4 - e](z_j) = 2z_3^2 z_4 + 2z_{10}$$

Suppose that we choose a pair of vertices $\{u, v\}$ in K_4 - e. We observe that there are three equivalence classes of choices, where the equivalence is induced by isomorphism of pairs

$$(K_4 - e, \{u, v\})$$

We will first consider u and v to be the two vertices of valence two, so that if we were to add an edge between them, we would obtain K_4. For this choice, the augmented imbedding polynomial is

$$2z_3(u_{**})z_3(v_{**})z_4(u_*v_*) + 2z_{10}(u_*v_{**}u_*v_{**})$$

The parenthetic expression following each variable shows the order in which the vertices u and v are encountered on a tour of the boundary of the corresponding face. The asterisks are used as wild cards to denote the occurence of any vertex other than u or v. In general, a single term of the unaugmented imbedding polynomial might split into several terms of the augmented imbedding polynomial. Splitting would occur if there were two imbeddings with the same number of faces of each size, but in which the relative locations of the special vertices difffered. However, a quick survey of the four imbeddings of

$K_4 - e$ indicates that no splitting occurs.

Since the first term of the augmented imbedding polynomial has two occurences of u and two occurences of $\bar{v}$, there are four (two times two) ways to imbed the additional edge in an imbedding of the augmented type it represents. If the additional edge runs from one 3-sided face to the other 3-sided face, then the resulting (unaugmented, of course) polynomial term is

$$z_4 z_8$$

because the two 3-sided faces plus the handle betwen them which is cut open by the additional edge form an 8-sided face, while the 4-sided face remains.

If the additional edge runs from a 3-sided face to the 4-sided face, which might happen in two different ways, the resulting polynomial term is

$$z_3{}^4$$

It follows, in summary, that the first term of the given augmented imbedding polynomial for $K_4 - e$ contributes

$$2z_3{}^4 + 4z_3 z_9 + 2z_4 z_8$$

to the imbedding polynomial for K_4.

Next let's consider the term $2z_{10}(u_\bullet v_{\bullet\bullet} u_\bullet v_{\bullet\bullet})$. There are two ways to imbed the additional edge so that the 10-sided face is split into a 3-sided face and a 9-sided face, and two ways so that it is split into a 4-sided face and an 8-sided face. Thus, that term of the augmented imbedding polynomial for $K_4 - e$ contributes

$$4z_3 z_9 + z_4 z_8$$

to the imbedding polynomial for K_4. When this contribution is added to the contribution of the previous term, the result is

$$2z_3{}^4 + 8z_3 z_9 + 6z_4 z_8$$

4 Imbedding distributions for bar-amalgamations

The augmented imbedding polynomials of the previous section are a first step toward a systematic theory of imbedding distributions, since they enable one to calculate the effect of adding an edge. In this section we define an operation that allows one to construct a new graph from two arbitrary existing graphs. We then show how algebraically to derive the genus polynomial and the imbedding polynomial of the resulting graph in terms of the polynomials for the constituent graphs. More generally, one would be interested in the effect of many different kinds of graphical composition on the various imbedding invariants.

In particular, we define the *bar-amalgamation* of two disjoint rooted graphs (G, u) and (H, v) to be the result of running a new edge (called the "bar") from the root vertex u of G to the root vertex v of H. Figure 4.1 shows a bar-amalgamation of two copies of K_4 - e, in which the root vertex u of the lefthand copy (called L) is 3-valent and the root vertex v of the righthand copy (called R) is 2-valent.
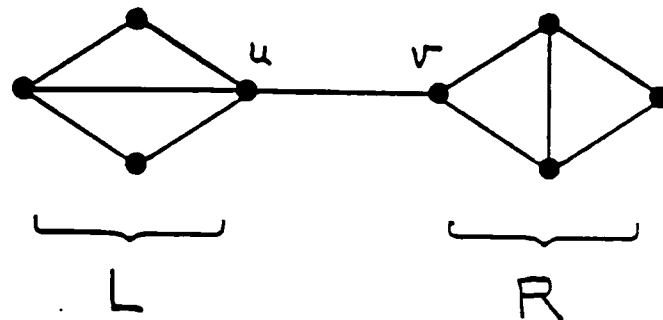

Figure 4.1 A bar-amalgamation of two copies of K_4 - e.

Theorem 4.1 The genus polynomial for the bar-amalgamation of the rooted graphs (G, u) and (H, v) is a constant multiple of the product of the genus polynomials for the graphs G and H. The constant factor equals the valence of u in G times the valence of v in H. (Equivalently, one might say that the genus distribution of the bar-amalgamation is a constant multiple of the convolution of the genus distributions of the component graphs.)

Proof: Consider any rotation systems for the graphs G and H. A rotation system for the bar-

amalgamation is obtained by inserting the vertex v somewhere into the G-rotation at u and inserting the vertex u somewhere into the H-rotation at v and then taking the union of the two adjusted systems. Moreover, every rotation system for the bar-amalgamation can be obtained in this manner.

Let's consider how to construct the imbedding of the bar-amalgamation that corresponds to that rotation system.

Suppose that the original G-rotation system corresponds to an imbedding of G into the surface S_r and that the original H-rotation system corresponds to an imbedding of H into the surface S_s. Suppose also that the vertex v was inserted between the G-vertices a and b of the rotation

$$u. \; ... \; a \; b \; ...$$

and that the vertex u was inserted between the H-vertices c and d of the rotation

$$v. \; ... \; c \; d \; ...$$

Imagine a surface formed by connecting the imbedding surface S_r for the graph G by a tube to the imbedding surface S_s for the graph H. The G-end of the tube is attached to the face of the G-imbedding that contains the sequence ... a u b ... on its boundary. The H-end of the tube is attached to the face of the H-imbedding that contains the sequence ... c v d ... on its boundary. Then the connecting bar is placed onto the surface so that it runs from the vertex u to the vertex v along this tube. The genus of this resulting surface is r + s.

QED

For instance, from Example 4 we know that the isomorphic graphs L and R of Figure 4.1 both have the genus polynomial

$$2 + 2x$$

The vertex u has valence 3 in the graph L and the vertex v has valence 2 in the graph R. Thus, the genus polynomial for the bar-amalgamation of (L, u) and (R, v) is

$$6(2 + 2x)(2 + 2x) = 24 + 48x + 24x^2$$

Routine analysis shows that the augmented polynomial for the rooted graph (L, u) is

$$2z_3(u_{\bullet\bullet})z_3(u_{\bullet\bullet})z_4(u_{\bullet\bullet\bullet}) \; + \; 2z_{10}(u_{\bullet\bullet\bullet}u_{\bullet\bullet}u_{\bullet\bullet})$$

and that the augmented polynomial for the rooted graph (R, v) is

$$2z_3(v_{\bullet\bullet})z_3z_4(v_{\bullet\bullet\bullet}) \; + \; 2z_{10}(v_{\bullet\bullet\bullet\bullet}v_{\bullet\bullet\bullet\bullet})$$

By an extension of the reasoning used in the proof of Theorem 4.1, we can now calculate the imbedding polynomial of the bar-amalgamation of (L, u) and (R, v), as follows.

Designate any instance of the root vertex u in an augmented L-term and any instance of the root vertex v—in an augmented R-term. For instance, we might designate the instance of u in the first occurrence of the variable z_3 in the first L-term

$$2z_3(\underline{u}_{\bullet\bullet})z_{3(u_{\bullet\bullet})}z_4(u_{\bullet\bullet\bullet})$$

and the instance of v in the variable z_4 of the first R-term

$$2z_3(v_{\bullet\bullet})z_3z_4(\underline{v}_{\bullet\bullet\bullet})$$

We have used underscoring to mark our choice of instances.

For each such choice of a pair of instances, there is a monomial contribution to the imbedding polynomial of the bar-amalgamation. First, the two designated variables combine to form a single variable in the resultant monomial, and that combined variable has as its subscript two plus the sum of the subscripts of the two designated variables. For instance, the u-variable z_3 of the L-term and the v-variable z_4 of the R-term combine to form the variable z_9, since $3 + 4 + 2 = 9$. The explanation is that when the bar is drawn on the tube from u to v, there is a face formed that contains all the sides of the previous u-face and all the sides of the previous v-face and both sides of the connecting bar. This combined variable is multiplied by all the remaining subscripted variables of the L-term and by all the remaining variables of the R-term, thereby to complete the calculation of the monomial contribution. It follows that the monomial contribution corresponding to our example choice of instances of u and v is

$$4z_3{}^3z_4z_9$$

The coefficient 4 is simply the product of the coefficients of the contributing L-term and R-term.

If we sum the contributions for all choices of instances of u and v in our example, and if we then collect similar monomials, then we obtain the following imbedding polynomial for the bar-amalgamation graph:

$$8z_3^2z_4^2z_8 \;+\; 12z_3^3z_4z_9 \;+\; 4z_3^4z_{10} \;+\; 28z_3z_4z_{15} \;+\; 20z_3^2z_{16} \;+\; 24z_{22}$$

5 The multiset of all duals

To every imbedding of a graph G in a surface S, there is a *dual graph* in the sense of Poincare, with a natural dual *imbedding*. The adjective "primal" refers to the original graph or to the original imbedding.

Starting with the primal graph imbedding, here is how to construct the dual graph. First place a vertex in the interior of every region of the primal imbedding. Then, through every primal edge, draw a dual edge from the vertex on one side of the primal edge to the vertex on the other. If both sides of a primal edge lie on the boundary of the same primal region, then both ends of the corresponding dual edge are incident on the single dual vertex that corresponds to that primal region.

By the *multiplicity* of a dual graph, we mean the number of different imbeddings of the primal graph for which it arises as the dual graph.

By the *multiset of duals* of a graph G, we mean the set of all duals and their multiplicities. Alternatively, we sometimes call this multiset the *dual space* of G.

Example 5 The graph with one vertex and n self-adjacencies, is called the *bouquet of n circles*, and it is denoted B_n. We will concentrate on the bouquet B_n. Since its one vertex has valence six, there are 120 (i.e., 5!) imbeddings. Two different dual graphs arise from imbeddings of B_3 in the sphere, and three additional dual graphs arise from imbeddings of B_3 in the torus. Figures 5.1 and 5.2 depict these five possible dual graphs, and the edge-labels show the multiplicity of each dual. Of course, the sum of the multiplicities is 120, the total number of imbeddings.
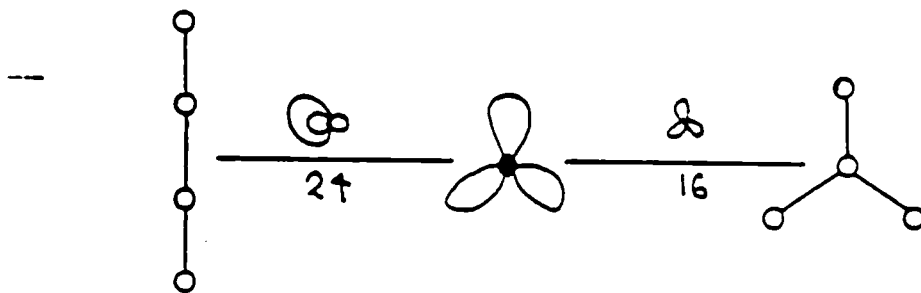

Figure 5.1   The bouquet B_3 has 40 imbeddings in the sphere, of which 24 have a 3-edge path for a dual graph, and of which 16 have the claw $K_{1,3}$.
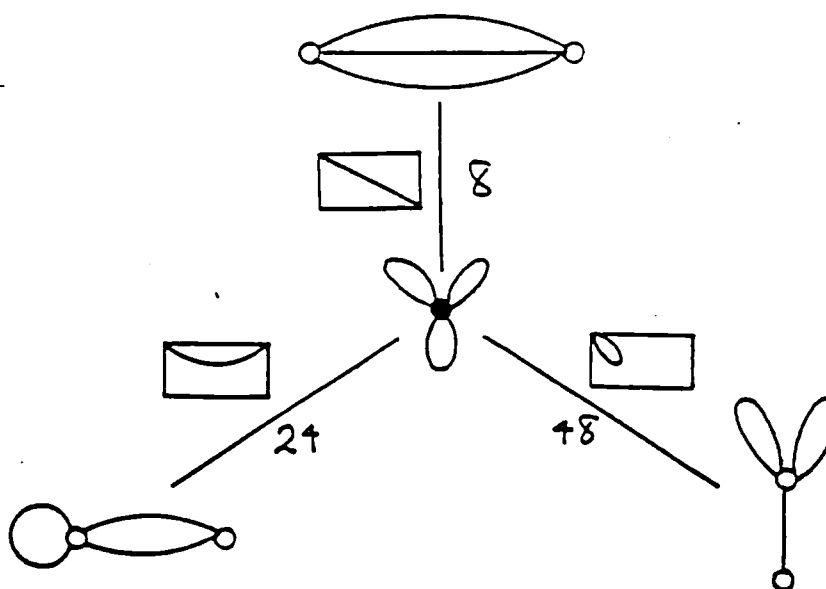
Figure 5.2 The bouquet B_3 has 80 imbeddings in the torus, for which there are three different dual graphs in all.

From each dual graph and its multiplicity, one can readily calculate a corresponding imbedding monomial and a coefficient. Thus, the multiset of duals for a graph G contains at least as much information as the imbedding polynomial. For instance, Figure 5.1 enables us to calculate the spherical part of the imbedding polynomial $i[B_3]$ as

$$24z_1^2z_2^2 + 16z_1^3z_3$$

and Figure 5.2 enables us to calculate the toroidal part as

$$48z_1z_5 + 24z_2z_4 + 8z_3^2$$

If two different dual graphs happened to have the same valence sequence, then their corresponding monomials would be the same, so that their contributions would be represented by a single term of the imbedding polynomial.

The relationship between the multiset of duals and the imbedding polynomial has considerable theoretical importance, as illustrated by its use in the proof of the following theorem.

Theorem 5.1 Let $h(n)$ be the maximum number of terms in the imbedding polynomial of any n-edge graph. Then $h(n)$ is an exponential function of the argument n.

Proof: It has long been known that the number $p(n)$ of partitions of the integer n is an exponential function of n. (For an exposition and an historical perspective, see Chapter 12 of Grosswald [1966].) It is little additional effort to show that the numbers of odd or even partitions of 2n with all parts equal to four or more is also an exponential function of n.

We would like to establish that if the number n is even, then every odd partition with each part equal to four or more is represented by a term of the imbedding polynomial of the bouquet B_n. Any n-edge graph with a one-face imbedding is a dual of the bouquet B_n. Of course, the valence sequence of a dual for B_n equals a given partition of the number 2n, then that partition is represented by a term of $i[B_n]$. Thus, it suffices to exhibit an n-edge graph with a one-face imbedding and a valence sequence equal to the given partition.

Suppose there are $2k + 1$ parts to the partition. Then we shall start with a graph J_{2k+1} that has $2k + 1$ vertices and valence four at every vertex. Simply double every edge of the (2k+1)-vertex path P_{2k+1} and add a self-adjacency at each end. (Furst, Gross, and Statman [1985] call such a graph a "cobblestone path".)

By taking the original path as a spanning tree in J_{2k+1}, we see that the span deficiency is zero, because there is only one component in the complement, and it is an even component with exactly $2k+2$ edges. Thus, the (4k+2)-edge cobblestone path J_{2k+1} is dual to the bouquet B_{4k+2}. Any partition of any even number larger than $4k + 2$ into $2k + 1$ parts with each parts equal to four or more can be represented by the valence sequence of a graph obtained by adding evenly many edges to J_{2k+1}. The addition of evenly many edges will preserve the evenness of the complement of the spanning tree, so that there will be a one-face imbedding.

Similarly, if the number n is odd, then every even partition of 2n with each part equal to four or more is represented by the valence sequence of a one-face imbeddable graph.

It follows that the number of terms of the imbedding polynomial $i[B_n]$ is an exponential function of n.

QED

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